Discrete Mathematics 21-228
Assignment #3 Solutions

1. Proof by induction on n:
   Base case ($n = 0$): $(x + y)^0 = 1 = \sum_{k=0}^{0} \binom{n}{k} x^k y^{n-k}$
   Inductive step: Assume the theorem is true for some $m \geq 0$, i.e.
$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}$$
   Then
$$\begin{aligned}
(x+y)^{(m+1)} &= (x+y)(x+y)^m \\
&= (x+y)\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} \text{ by inductive hypothesis} \\
&= x\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} + y\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} \\
&= \sum_{k=0}^{m}\binom{m}{k}x^{k+1}y^{m-k} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k+1} \\
&= \sum_{j=1}^{m+1}\binom{m}{j-1}x^j y^{m-j+1} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k+1} \\
&= x^{m+1} + \sum_{j=1}^{m}\binom{m}{j-1}x^j y^{m-j+1} + \sum_{k=1}^{m}\binom{m}{k}x^k y^{m-k+1} + y^{m+1} \\
&= x^{m+1} + \sum_{k=1}^{m}(\binom{m}{k-1} + \binom{m}{k})x^k y^{(m+1)-k} + y^{m+1} \\
&= x^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}x^k y^{(m+1)-k} + y^{m+1} \text{ by Pascal} \\
&= \sum_{k=0}^{m+1}\binom{m+1}{k}x^k y^{(m+1)-k}
\end{aligned}$$

2. By induction on n:
   Base case ($n = 1$): $F_2 = 1 = F_3 - 1$.
   Inductive step: Assume true for some $m \geq 1$, i.e.
$$\sum_{i=1}^{m} F_{2i} = F_{2m+1} - 1$$
   We want to show:
$$\sum_{i=1}^{m+1} F_{2i} = F_{2(m+1)+1} - 1 = F_{2m+3} - 1$$

$$
\begin{aligned}
\sum_{i=1}^{m+1} F_{2i} &= \sum_{i=1}^{m} F_{2i} + F_{2(m+1)} \\
&= \sum_{i=1}^{m} F_{2i} + F_{2m+2} \\
&= F_{2m+1} - 1 + F_{2m+2} \text{ by ind. hypothesis} \\
&= F_{2m+3} - 1
\end{aligned}
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3. Count grid paths recursively as in class. For the first diagram, there will be $\binom{10}{5}$ grid paths. For the second diagram, there will be a total of 48 paths.

4. In expanding $(x+y+z)^{11}$, we get a $x^3y^6z^2$ term every time we multiply a string of 3 x's, 6 y's and 2 z's. There are $\binom{11}{3,6,2} = \frac{11!}{3!6!2!}$ such strings. (This is where the term *multinomial coefficient* comes from.)

5. Say Bob is one of the people at the party. There are 5 other people at the party each of whom is a friend of Bob or an enemy of Bob. By the Pigeonhole Principle, Bob has at least three friends at the party or 3 enemies. If Bob has at least three friends, pick a group of 3 of them. If two of these people are friends (say Alice and Jim), then Bob, Alice, and Jim are a group of three friends. Otherwise, these 3 friends of Bob are a group of 3 enemies. If Bob has 3 enemies, the argument is exactly the same.
   In a party of 5 people, arrange the people around a circular table. If each person is an enemy of their two immediate neighbors, but a friend of the others, then there is no group of 3 friends or enemies.