Discrete Mathematics 21-228
Assignment #2 Solutions

1. (a) $\binom{A\cup B}{2} = \binom{A}{2} \cup \binom{B}{2}$
False: If $A = \{1\}, B = \{2\}$, then $\binom{A\cup B}{2} = \{\{1,2\}\}$, but $\binom{A}{2} \cup \binom{B}{2} = \emptyset \cup \emptyset = \emptyset$.
   (b) $\binom{A\cup B}{2} \supseteq \binom{A}{2} \cup \binom{B}{2}$
True: Any 2-subset of A or B is also a 2-subset of $A \cup B$.
   (c) $\binom{A\cap B}{2} = \binom{A}{2} \cap \binom{B}{2}$
True: If C is a 2-subset of $A \cap B$, then it must be a 2-subset of both A and B. Conversely, if C is a 2-subset of A and B, then it must be a subset of $A \cap B$.
   (d) $\binom{A\cap B}{2} \subseteq \binom{A}{2} \cap \binom{B}{2}$
True: These two sets are identical, thus they are each subsets of each other.

2. We count 20-digit strings: There are 10 choices for the first digit. Then there are nine choices for each successive digit, since we may choose any digit except the one we just chose. Thus there are $10 \cdot 9^{19}$ possible twenty-digit strings.
If we exclude strings that begin with zero, then there are only nine choices for the first digit as well, so there are a total of 9^{20} 20-digit integers.
Either of the above answers is an acceptable interpretation of the problem.

3. There are $\binom{23}{12} = \frac{23!}{12!11!}$ ways for the craftsman to choose the 12 beads. Once these beads are chosen, there are $12!/(12\cdot 2) = 11!/2$ ways to arrange them on the bracelet. Why? If the beads were just arranged in a line, there would be 12! possible orderings, but since they are arranged in a circle, any particular line ordering will appear identical to any other line ordering that can be obtained by rotating it (E.g. the line orderings $\{1, 2, 3, \ldots, 12\}$ and $\{2, 3, \ldots, 12, 1\}$ are indistinguishable. There are twelve different ways to rotate a given line ordering, so counting line orderings overcounts by a factor of 12. We must also divide by 2 since the bracelet may be flipped over, and so any ordering will be indistinguishable from its mirror ordering (e.g. $\{1, 2, \ldots, 12\}$ is indistinguishable from $\{12, 11, \ldots, 1\}$.
Thus, in total, there are

$$\frac{23!}{12!11!} \cdot \frac{11!}{2} = \frac{23!}{12! \cdot 2}$$

ways to arrange 12 of the 23 beads on a bracelet.

4. The number of strings is $\binom{q+1}{p}$. Lay down the q zeros in a line. Then there are $q + 1$ positions where we can place a 1: the $q - 1$ gaps between the 0's, and the 2 positions before or after the 0's. Since no two 1's may be adjacent, we must choose exactly p of these positions to contain 1's.

5. Let A be the set of k-subsets of $[n]$. $|A| = \binom{n}{k}$.

   - Let $A_{1,2}$ be the set of k-subsets of $[n]$ that do not contain 1 or 2. $|A_{1,2}| = \binom{n-2}{k}$.
   - Let A_2 be the set of k-subsets of $[n]$ that do not contain 1, but do contain 2. $|A_2| = \binom{n-1}{k-1}$
   - Let A_1 be the set of k-subsets of $[n]$ that do not contain 2, but do contain 1. $|A_1| = \binom{n-1}{k-1}$
   - Let A_0 be the set of k-subsets of $[n]$ that contain both 1 and 2. $|A_0| = \binom{n-2}{k-2}$

   A_0, A_1, A_2, and $A_{1,2}$ are a partition of A, and thus $|A| = |A_0| + |A_1| + |A_2| + |A_{1,2}|$, which proves the statement.