Discrete Mathematics 21-228
More Review Problems

The best sources of review material for the final are your class notes, homeworks, and exams. Below are a few problems if you wish to practice. Some of these problems may be harder than those on the final, some easier.

1. How many labeled trees on 10 vertices have exactly 3 leaves?
2. How many tournaments are there on n labeled vertices?
3. Prove: Every tree is bipartite.
4. Solve the recurence
$$a_n - 7a_{n-1} + 12a_{n-2} = 0$$
for $n \geq 2$, with $a_0 = a_1 = 1$.
5. Let p_n denote the number of permutations which do not have cycles of length 2. Find a recurrence for p_n.
6. Show that the number of binary sequences which do not contain any subsequence of the form 101 satisfies the recurrence
$$c_n = c_{n-1} + c_{n-3} + c_{n-4} + \cdots + c_0 + 2$$
for $n \geq 3$, with $c_0 = 1, c_1 = 2, c_2 = 4$. Can you find an order k recurrence for this sequence?
7. Let $B(n,k)$ be the number of partitions of $[n]$ into k non-empty parts, each of which is of *odd* cardinality. Find a recurrence for $B(n,k)$.
8. How many pairs of permutations π_1, π_2 of $[n]$ are there for which there is no i such that $\pi_1(i) = \pi_2(i) = i$?
9. Let T_1 and T_2 be spanning trees of a graph G and let e be an edge of T_1 not in T_2. Show that there is a spanning tree T_3 of G containing e and all but one edge from T_2.
10. Two people play a game on a graph G by alternately selecting distinct vertices $v_0, v_1, \ldots$, such that for $i > 0$, v_i is adjacent to v_{i-1}. The last player able to select such a vertex wins. Show that the first player has a winning strategy if and only if the graph does not have a perfect matching.
11. Let $n < m$ be positive integers. Let H be a collection of n-element subsets of $[m]$. Suppose the number of sets in H is less than $4^{n-1}/3^n$. Show that there exists a coloring of $[m]$ with four colors such that there is a color C such that every set in H has at least one element colored C. Hint choose a coloring at random and use Boole's inequality.
12. What is the probability that the top and bottom cards of a randomly shuffled deck are both aces?

13. Suppose we color the edge set of K_n at random with two colors. Let X be the number of monochromatic triangles in the random coloring. Determine $\mathbf{E}(X)$. Conclude that there exists a coloring of the edges of K_n that has at most $\frac{1}{4}\binom{n}{3}$ monochromatic triangles.

14. Let G be a graph in which every vertex has degree at least 2. Show that G contains a cycle.

15. Consider the cycle C_4 as representing a communication network among 4 nodes $\{a, b, c, d\}$. Suppose each communication line fails with probability p (i.e. each edge is absent with probability p). Let A be the event that b is connected ot d. Let B be the event that a is connected to c. Compute $Pr(A)$ and $Pr(A|B)$. Are A and B independent events?

16. Find the number of integers between 1 and 10,000 that are neither perfect squares or perfect cubes.