Discrete Mathematics 21-228
Final Exam Solutions

You may refer to any class materials (notes, textbook, references listed on the syllabus or course website, homeworks and exams), but no other materials. In particular you should not discuss these problems with anyone but me. Send me email if you have any questions. I will check it frequently. The exam is due at 2:50 in DH 2105. Good luck!

Show your work and justify your results.

The *n-cube*, denoted by Q_n, is the graph that has vertices representing all bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

1. Draw Q_1, Q_2, Q_3, and Q_4.

2. Prove that Q_n has a Hamilton cycle for all $n \geq 2$. (Hint: use induction)
   Base case: If $n = 2$, then Q_n is a 4-cycle, and is thus Hamiltonian.
   Inductive step: Assume true for all $k < n$. Then we can construct a Hamilton cycle for Q_n as follows: The vertices of Q_n can be partitioned into two sets A and B as follows: Let A be those strings whose first bit is 0 and B be those whose first bit is 1. The subgraphs induced by A and B each have a Hamilton cycle by the inductive hypothesis, since each is isomorphic to Q_{n-1}. These two cycles may be spliced together to form a Hamilton cycle for Q_n.

3. For which values of n does Q_n have an Euler Tour?
   Q_n is an n-regular graph, and thus has an Euler Tour if and only if n is even.

4. How many vertices and edges does Q_n have?
   Q_n has 2^n vertices. Since each vertex has degree n,
   $$|E| = \frac{|V|n}{2} = n2^{n-1}$$

5. What is $\chi(Q_n)$? Describe a coloring using the minimum number of colors.
   $\chi(Q_n) = 2$. We can color all strings with an even number of 1's red, and all others blue. Note that since Q_n is connected, this is the only possible bipartition.

6. For which values of n is Q_n planar?
   Euler's formula $\Rightarrow$ Q_n is not planar if $n \geq 6$. By methods not discussed in class, it can be shown that Q_n is planar if and only if $n \leq 3$.

7. Prove that Q_n has a perfect matching for all n.
   Again by induction on n: Base case: If $n = 1$, then the only edge is a

perfect matching.
Inductive step: Assume true for all $k < n$. As in question 2, partition the vertices into two parts, each with an induced subgraph isomorphic to Q_{n-1}. By the inductive hypothesis, each of these parts has a perfect matching, and thus the whole graph has a perfect matching.

8. How many perfect matchings does Q_n have?
Many. If a_n is the number of perfect matchings of Q_n, then a lower bound on a_n is the sequence that satisfies the recurrence

$$a_n = n a_{n-1}^2$$

with $a_1 = 1$.

9. How many spanning trees does Q_n have?
Many. If a_n represents the number of spanning trees of Q_n then a lower bound on a_n is the sequence satisfying the recurrence

$$a_n = 2^{n-1} a_{n-1}^2$$

, with $a_1 = 1$.

10. Given two vertices chosen uniformly at random (possibly the same), what is the probability they are different endpoints of an edge?
There are $(2^n)^2 = 2^{2n}$ ways to pick two vertices, and $n2^{n-1}$ edges. Each edge could have its endpoints picked two different ways, and thus

$$Pr(endpoints) = \frac{2n2^{n-1}}{2^{2n}} = \frac{n}{2^n}$$

Alternatively, two vertices are endpoints of the same edge if their bit strings differ in exactly one position. There are n possible positions that they could differ in, and the probability that 2 strings agree or disagree in any given position is $1/2$, and thus,

$$Pr(endpoints) = n\left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{n-1} = n\left(\frac{1}{2}\right)^n = \frac{n}{2^n}$$

11. Let the *distance* $d(u, v)$ between any two vertices be the length of the shortest path between u and v. Describe which pairs of vertices have the greatest distance from each other in Q_n. What is this distance? For each of these pairs, how many such paths are there?
The pairs of vertices with greatest distance are those which disagree at every bit position (there are 2^{n-1} such pairs). The shortest path between such pairs has length n. Each of these paths can be thought of as sequentially flipping one of the n bits to take one step closer to the destination. The bits can be flipped in any order, and thus there are $n!$ paths of length n between such pairs of points.

12. Given two vertices chosen uniformly at random, what is the expected distance between them?

The expected distance is equal to the expected number of bits on which the two strings disagree. The probability that two strings agree or disagree on any given bit is 1/2. Thus the probability that two strings disagree on exactly k bits is

$$\binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \frac{1}{2^n}\binom{n}{k}$$

Thus the expected distance between two vertices is

$$\begin{aligned}
\sum_{k=0}^{n} k \cdot \frac{1}{2^n}\binom{n}{k} &= \frac{1}{2^n}\left(\sum_{k=1}^{n} k\binom{n}{k}\right) \\
&= \frac{1}{2^n}\left(n\sum_{k=0}^{n-1}\binom{n-1}{k}\right) \qquad \text{(think of picking a committee, then a president)} \\
&= \frac{1}{2^n}(n2^{n-1}) \\
&= \frac{n}{2}
\end{aligned}$$

Alternatively, let the r.v. X denote the distance between the two vertices, and let $X_i = 1$ if the two vertices disagree at bit i, 0 otherwise. Then $\mathbf{E}(X_i) = \frac{1}{2}$ for each i, and $X = \sum X_i$, so by linearity of expectation,

$$\mathbf{E}(X) = \sum_{i=1}^{n} \mathbf{E}(X_i) = n \cdot \frac{1}{2} = \frac{n}{2}$$

13. Assume n is even. Let S be the set of vertices representing strings with an equal number of 1's and 0's. How big is $|S|$? Use the Law of Large Numbers (version proved in class) to put an upper bound on the probability that a randomly selected vertex is at distance more than $n/2^6$ from S.

$|S| = \binom{n}{n/2}$.

We can think of a randomly chosen string as the outcome of n coin flips (0=H, 1=T, for instance). Thus we get a binomial distribution on the probability of getting a certain number of 0's. Note that a point is at distance more than $n/2^6$ from S if it has more than $(n/2 + n/2^5)$ 1's or 0's. The version of the law of large numbers we derived in class stated that

$$Pr(|B_{n,p} - np| \geq \epsilon np) \leq \frac{1}{\epsilon^2 np}$$

Substituting $p = 1/2, \epsilon = 1/2^4$, we get

$$Pr(|B_{n,p} - n/2| \geq n/2^5) \leq \frac{2^9}{n}$$

and thus

$$Pr(\text{a point is at distance greater than } n/2^6) \leq \frac{2^9}{n}$$

14. Assume n is even, and define S as above. Given that two vertices are chosen uniformly at random from S, what is their expected distance?
Given a vertex $v \in S$, a second vertex $u \in S$ can agree on exactly $2k$ bits in $\binom{n/2}{k}^2$ ways. There are $\binom{n}{n/2}$ ways to pick u. Thus

$$Pr(u \text{ agrees on 2k bits}) = \frac{\binom{n/2}{k}^2}{\binom{n}{n/2}}$$

If two vertices agree on $2k$ bits, then the distance between them is $n-2k$, and so the expected distance between u and v is

$$\begin{aligned}
\sum_{k=0}^{n/2}(n-2k)\frac{\binom{n/2}{k}^2}{\binom{n}{n/2}} &= \frac{1}{\binom{n}{n/2}}\left(n\sum_{k=0}^{n/2}\binom{n/2}{k}^2 - 2\sum_{k=0}^{n/2}k\binom{n/2}{k}^2\right) \\
&= \frac{1}{\binom{n}{n/2}}\left(n\binom{n}{n/2} - 2\sum_{k=1}^{n/2}k\binom{n/2}{k}^2\right) \\
&= \frac{1}{\binom{n}{n/2}}\left(n\binom{n}{n/2} - 2(n/2)\binom{n-1}{n/2-1}\right) \\
&= n - n/2 \\
&= n/2
\end{aligned}$$

The steps used to transform the sums of binomial coeffiecients can be justified combinatorially by again thinking of counting ways to pick committees and presidents.
Alternatively, given a vertex $v \in S$, for every vertex in S at distance k, its' "opposite" (i.e. vertex which disagrees on all bits) will also be in S, and be at distance $(n-k)$. These pairs' average distance from v is $n/2$, and thus summing over all pairs, we see that the average distance from v is $n/2$.

15. Color Q_n randomly using 3 colors. What is the expected number of squares (i.e. 4-cycles) which have all vertices the same color? Label two squares A and B. Are the events $\{A \text{ is monochromatic}\}$ and $\{B \text{ is monochromatic}\}$ independent events?
There are $\binom{n}{2}2^{n-2}$ squares in Q_n ($\binom{n}{2}$ ways to bick two bits to flip for the 4-cycle, and 2^{n-2} ways to choose the other $n-2$ bits). Represent this quantity by m. Let the r.v. X count the number of monochromatic squares, $X_i = 1$ if square i is monochromatic, 0 otherwise. Then $\mathbf{E}(X_i) =$

$\frac{1}{27}$ for each i, and $X = \sum X_i$. Thus, by linearity of expectation,

$$\mathbf{E}(X) = \sum_{i=1}^{m} \mathbf{E}(X_i) = m \cdot \frac{1}{27} = \frac{m}{27}$$

$$\Rightarrow \mathbf{E}(X) = \frac{\binom{n}{2} 2^{n-2}}{27}$$

The events $\{A$ is monochromatic$\}$ and $\{B$ is monochromatic$\}$ are independent if and only if A and B share no vertices.