Discrete Mathematics 21-228
Final Exam

You may refer to any class materials (notes, textbook, references listed on the syllabus or course website, homeworks and exams), but no other materials. In particular you should not discuss these problems with anyone but me. Send me email if you have any questions. I will check it frequently. The exam is due at 2:50 in DH 2105. Good luck!

Show your work and justify your results.

The *n-cube*, denoted by Q_n, is the graph that has vertices representing all bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

1. Draw Q_1, Q_2, Q_3, and Q_4.
2. Prove that Q_n has a Hamilton cycle for all $n \geq 2$. (Hint: use induction)
3. For which values of n does Q_n have an Euler Tour?
4. How many vertices and edges does Q_n have?
5. What is $\chi(Q_n)$? Describe a coloring using the minimum number of colors.
6. For which values of n is Q_n planar?
7. Prove that Q_n has a perfect matching for all n.
8. How many perfect matchings does Q_n have?
9. How many spanning trees does Q_n have?
10. Given two vertices chosen uniformly at random (possibly the same), what is the probability they are different endpoints of an edge?
11. Let the *distance* $d(u, v)$ between any two vertices be the length of the shortest path between u and v. Describe which pairs of vertices have the greatest distance from each other in Q_n. What is this distance? For each of these pairs, how many such paths are there?
12. Given two vertices chosen uniformly at random, what is the expected distance between them?
13. Assume n is even. Let S be the set of vertices representing strings with an equal number of 1's and 0's. How big is $|S|$? Use the Law of Large Numbers (version proved in class) to put an upper bound on the probability that a randomly selected vertex is at distance more than $n/2^6$ from S.
14. Assume n is even, and define S as above. Given that two vertices are chosen uniformly at random from S, what is their expected distance?

15. Color Q_n randomly using 3 colors. What is the expected number of squares (i.e. 4-cycles) which have all vertices the same color? Label two squares A and B. Are the events $\{A \text{ is monochromatic}\}$ and $\{B \text{ is monochromatic}\}$ independent events?