Discrete Mathematics 21-228
Exam #4
Tuesday, June 22

You may use your notes and textbook. Show your work, and justify your results.

1. A *vertex cover* of a graph $G = (V, E)$ is a set $S \subseteq V$ that contains at least one endpoint of every edge.

   (a) What is the minimum size of a vertex cover for the following graphs: $P_n, C_n, K_n, K_{m,n}$?

   (b) Prove that if S is any vertex cover of G, and M is any matching of G, then $|S| \geq |M|$.

   (a) The minimum vertex cover for P_n is $n/2$ if n is even, $(n-1)/2$ if n is odd.
   The minimum vertex cover for C_n is $n/2$ if n is even, $(n+1)/2$ if n is odd.
   The minimum vertex cover for K_n is $n-1$ if $n \geq 2$, 1 if $n = 1$.
   The minimum vertex cover for $K_{m,n}$ is $min\{m, n\}$.

   (b) S must contain a vertex adjacent to every edge in M, and no vertex is adjacent to more than one edge in M, so at least $|M|$ vertices are required just to cover the edges of M.

2. Let $T = (V, E)$ be a labeled tree on 8 vertices.

   (a) How many such trees are there?

   (b) Prove that the degree of a vertex $v \in V$ is one greater than the number of times v's label appears in the Prufer code for T.

   (c) Given the Prufer code for T, explain how you could tell whether T is a path.

   (d) Given the Prufer code for T, explain how you could tell whether T is a star.

   (e) How many labeled paths on 8 vertices are there?

   (f) How may labeled stars on 8 vertices are there?

   (a) By Cayley's theorem, there are 8^6 such trees.

   (b) The label for v appears in the code once for every time a neighbor is deleted, until either v is deleted or left as the final two vertices. Since, v is only deleted when it has degree 1, in either case above, v's label appears in the code once for all but one of its neighbors.

   (c) A path on 8 vertices has 6 vertices of degree 2 and 2 vertices of degree 1. Thus the Prufer code for a path consists of 6 distinct labels. Conversely, a code with 6 distinct labels corresponds to a tree with exactly two leaves, i.e. a path.

(d) T is a star if and only if one vertex has degree 7, which happens if and only if the Prufer code consists of one label repeated 6 times.

(e) There are $\frac{8!}{2!}$ strings of six distinct digits chosen from [8], thus by part (c), there are $\frac{8!}{2!}$ paths.

(f) By part (d), there are 8 stars.

3. **Defintion:** A *complete k-partite graph* $K_{n_1,n_2,\ldots,n_k}$ has vertices partitioned into k subsets of size $n_1, n_2, \ldots, n_k$ elements each, with vertices adjacent if and only if they are in different subsets of the partition.

(a) For what values of n_1, n_2 does K_{n_1,n_2} have an Euler Tour?

(b) Draw $K_{1,2,3}$ and $K_{2,2,2}$.

(c) What is $\chi(K_{n_1,n_2,\ldots,n_k})$? Given a set of colors, how many different colorings are there?

(d) For what values of $n_1, n_2, \ldots, n_k$ does $K_{n_1,n_2,\ldots,n_k}$ have an Euler Tour?

(e) For what values of $n_1, n_2, \ldots, n_k$ does $K_{n_1,n_2,\ldots,n_k}$ have a Hamilton Cycle?

(a) Since an Euler Tour exists if and only if every vertex has even degree, K_{n_1,n_2} has an euler tour if and only if n_1 and n_2 are both even.

(b)

(c) $\chi(K_{n_1,n_2,\ldots,n_k}) = k$, and there are $k!$ colorings.

(d) $K_{n_1,n_2,\ldots,n_k}$ has an Euler Tour if and only if n_i is even for every i or n_i is odd for every i and k is also odd.
Proof: ($\Leftarrow$) In both of the cases above, it is clear that every vertex will have even degree.
($\Rightarrow$) Assume $K_{n_1,n_2,\ldots,n_k}$ has an Euler Tour. Then $deg(v)$ is even for every vertex v.

$$\Rightarrow \sum_{i \neq j} n_i \text{ is even for each } j$$

$$\Rightarrow n_i, n_j \text{ have the same parity for all } i, j.$$

Thus n_i is even for all i or odd for all i. If it is odd for all i, then k must be odd for the degree condition to hold.

(e) $K_{n_1,n_2,\ldots,n_k}$ has a Hamilton Cycle if and only if $\frac{n_i}{\sum n_i} \leq \frac{1}{2}$. (i.e. if no part contains more than half the vertices).
Proof: $K_{n_1,n_2,\ldots,n_k}$ has a Hamilton Cycle if and only if we can list all vertices so that no two vertices in the same part are listed cosecutively. Assume $n_1 \geq n_i$ for all i.
($\Rightarrow$) If $n_1 > \frac{1}{2}\sum n_i$, then such a list is impossible, by the pigeonhole principle.
($\Leftarrow$) Assume $n_1 \leq \frac{1}{2}\sum n_i$, and create such a list as follows: Create n_1

sublists, with an element of the first part at the head of each sublist. Take all other vertices and place them sequentially in the sublists as follows: First place teh vertices from the second part, then the third, etc., placing one vertex in the first list, then one in the second list, etc., returning to the first list after the n_1^{th} list. Note that two vertices from the same part are never placed in the same list, since there are at least as many lists as the size of any part. Note also that every list will contain at least two elements since the first part contains fewer than half of the vertices. Now concatenate all of the lists to form one long list. Note that no two vertices of the first part are adjacent (since there are at least 2 vertices in every list) and no two vertices of any other part are adjacent (since there is a vertex from the first part between them). Thus we have a list of all vertices with no two vertices from the same part adjacent, and this determines a Hamilton cycle.