Discrete Mathematics 21-228
Exam #3 Solutions

First note that

$$\begin{aligned}
Pr(\text{R}) &= Pr(\text{R}|\text{Urn 1}) + Pr(\text{R}|\text{Urn 2}) + Pr(\text{R}|\text{Urn 3}) \\
&= \frac{1}{3}(\frac{1}{3} + \frac{2}{3} + 0) \\
&= \frac{1}{3} \\
Pr(\text{B}) &= Pr(\text{B}|\text{Urn 1}) + Pr(\text{B}|\text{Urn 2}) + Pr(\text{B}|\text{Urn 3}) \\
&= \frac{1}{3}(\frac{1}{3} + \frac{1}{6} + \frac{2}{3}) \\
&= \frac{7}{18} \\
Pr(\text{G}) &= Pr(\text{G}|\text{Urn 1}) + Pr(\text{G}|\text{Urn 2}) + Pr(\text{G}|\text{Urn 3}) \\
&= \frac{1}{3}(\frac{1}{3} + \frac{1}{6} + \frac{1}{3}) \\
&= \frac{5}{18}
\end{aligned}$$

1. (a) $Pr(\text{G}) = \frac{5}{18}$

   (b) $Pr(\text{Urn 1}) = \frac{1}{3}$, $Pr(\text{R}) = \frac{1}{3}$, and $Pr(\text{Urn 1} \cap \text{R}) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. Thus, $Pr(\text{Urn 1}) \cdot Pr(\text{R}) = Pr(\text{Urn 1} \cap \text{R}) \Rightarrow$ These events are independent.

   (c)
$$\begin{aligned}
Pr(\text{G}|\text{Urn 2}) &= \frac{Pr(\text{G} \cap \text{Urn 2})}{Pr(\text{Urn 2})} \\
&= \frac{\frac{1}{3} \cdot \frac{1}{6}}{\frac{1}{3}} = \frac{1}{6}
\end{aligned}$$

   (d)
$$\begin{aligned}
Pr(\text{Urn 1}|\text{B}) &= \frac{Pr(\text{Urn 1} \cap \text{B})}{Pr(\text{B})} \\
&= \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{7}{18}} = \frac{2}{7}
\end{aligned}$$

2. (a) $Pr(\text{first R on fifth draw}) = (\frac{2}{3})^4 \cdot \frac{1}{3}$

   (b) $Pr(\text{25 R,25 B, 25 G}) = \binom{75}{25}\binom{50}{25}\left(\frac{1}{3}\right)^{25}\left(\frac{5}{18}\right)^{25}\left(\frac{7}{18}\right)^{25}$

   (c) In n Bernoulli Trials, each with probability of success p, we expect np successes. In this experiment, there are 75 trials, each with a probability of 1/3 of success (i.e. R) $\Rightarrow$ We expect $75 \cdot \frac{1}{3} = 25$ red balls.

(d) Denote by the random variable Y the number of "repeats." Let

$$Y_i = \begin{cases} 1 & \text{if ball } i \text{ is a repeat} \\ 0 & \text{otherwise} \end{cases}$$

Note

$$Y = \sum_{i=2}^{75} Y_i$$

and for each i,

$$\mathbf{E}(Y_i) = \left(\frac{1}{3}\right)^2 + \left(\frac{5}{18}\right)^2 + \left(\frac{7}{18}\right)^2$$

since for a ball to be a repeat, both it and the previous ball must be the same color. Thus by linearity of expectation,

$$\begin{aligned} \mathbf{E}(Y) &= \mathbf{E}\left(\sum_{i=2}^{75} Y_i\right) \\ &= \sum_{i=2}^{75} \mathbf{E}(Y_i) \\ &= 74\left(\left(\frac{1}{3}\right)^2 + \left(\frac{5}{18}\right)^2 + \left(\frac{7}{18}\right)^2\right) \end{aligned}$$