Discrete Mathematics 21-228
Exam #2 Solutions

1. We select 38 even positive integers, each less than 1000. Prove that there are two of them whose difference is at most 26.

   This is true by the pigeonhole principle: If the minimum difference were 28, then the minimum difference between the smallest and largest chosen numbers would be $37 \cdot 28 = 1036$. This is impossible since the humbers are all between 1 and 1000.

2. Solve the following recurrence:

$$g_{n+1} = -2g_n + 3g_{n-1}$$

   for $n \geq 1$, $g_0 = 5$, $g_1 = -3$.

   The characteristic polynomial for this recurrence is

$$q^2 + 2q - 3 = (q+3)(q-1)$$

   Thus the characteristic roots are $q_1 = -3$ and $q_2 = 1$, and the general solution is of the form

$$g_n = b_1^n + b_2(-3)^n$$

   The initial conditions give the system of equations:

$$b_1 + b_2 = 5$$
$$b_1 - 3b_2 = -3$$

   whose solutions are $b_1 = 3, b_2 = 2$. Thus,

$$g_n = 3 + 2(-3)^n$$

3. Let $a_n$ = the number of different ways to tile a 3 x $n$ grid of squares using 3 x 1 tiles.

   (a) Find a recurrence relation for $a_n$. State which values of $n$ your recurrence relation is valid for, and give the necessary initial conditions to determine the sequence. List the first six values of the sequence.

   (b) Find the generating function for $a_n$. Explain how you could determine the values of $a_n$ using this generating function (you need not sctually do this).

   (a) The recurrence relation is

$$a_n = a_{n-1} + a_{n-3}$$

   for $n \geq 3$, with initial conditions $a_0 = 1, a_1 = 1, a_2 = 1$.

   Beginning with $a_0$, the first six values of the sequence are 1, 1, 1, 2, 3, 4.

(b) Let

$$f(x) = \sum_{n\geq 0} a_n x^n$$

be the generating function for $a_n$.
We multiply the recurrence relation by $a_n$ and sum over all values of $n$ for which the recurrence is valid to get

$$\sum_{n\geq 3} a_n x^n = \sum_{n\geq 3} a_{n-1} x^n + \sum_{n\geq 3} a_{n-3} x^n$$

On the LHS, we get

$$\begin{aligned}\sum_{n\geq 3} a_n x^n &= f(x) - (a_0 + a_1 x + a_2 x^2)\\ &= f(x) - (1 + x + x^2)\end{aligned}$$

On the RHS, we get

$$\begin{aligned}\sum_{n\geq 3} a_{n-1} x^n + \sum_{n\geq 3} a_{n-3} x^n &= x \sum_{n\geq 3} a_{n-1} x^{n-1} + x^3 \sum_{n\geq 3} a_{n-3} x^{n-3}\\ &= x \sum_{n\geq 2} a_n x^n + x^3 \sum_{n\geq 0} a_n x^n\\ &= x(f(x) - (1 + x)) + x^3 f(x)\\ &= (x + x^3) f(x) - x - x^2\end{aligned}$$

So we get

$$\begin{aligned}f(x) - (1 + x + x^2) &= (x + x^3) f(x) - x - x^2\\ \Rightarrow f(x) - (x + x^3) f(x) &= 1 + x + x^2 - x - x^2\\ \Rightarrow (1 - x - x^3) f(x) &= 1\\ \Rightarrow f(x) &= \frac{1}{1 - x - x^3}\end{aligned}$$

To determine the values of $a_n$, we would look at the power series corresponding to this generating function. The coefficient of $x^n$ in this series would be the value of $a_n$.