Discrete Mathematics 21-228
Exam #1 Solutions

1. Twenty-four indistinguishable balls are placed in six boxes.

   (a) In how many different ways can this be done?
   If each box is assigned one of six colors, we can think of putting a ball in a box as painting it one of 6 colors. Thus there are $\binom{24+6-1}{6-1} = \binom{29}{5}$ ways.

   (b) In how many ways can this be done if each box must receive at least two balls?
   Put two balls in each box. This leaves 12 balls to distribute, which can be done in $\binom{12+6-1}{6-1} = \binom{17}{5}$ ways.

2. Twenty-four distinguishable balls are to be placed in six boxes.

   (a) In how many ways can this be done? There are six choices for each of the 24 balls,and since each ball is distinguishable, every different choice yields a different outcome. Thus there are 6^{24} ways.

   (b) In how many ways can this be done if box 1 must get 12 balls, box 2 must get 7 balls, box 3 must get 5 balls, and boxes 4, 5, and 6 must be empty?
   There are $\binom{24}{12}$ ways to put 12 balls in box 1, and then $\binom{12}{7}$ ways to put 7 of the remaining 12 balls into box 2. All remaining balls must be placed in box 3. Thus there are $\binom{24}{12}\binom{12}{7}$ ways. Note that this quantity is equal to $\frac{24!}{12!7!5!}$ and is sometimes denoted by the multinomial coefficient $\binom{24}{12,7,5}$. Multinomial coefficients are a generalization of a binomial coefficients.

3. Prove

$$\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$$

$$\begin{aligned}
\binom{n+1}{k} &= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \frac{n+1}{k}\frac{n!}{(k-1)!(n+1-k)!} \\
&= \frac{n+1}{k}\frac{n!}{(k-1)!(n-(k-1))!} \\
&= \frac{n+1}{k}\binom{n}{k-1}
\end{aligned}$$

4. Prove

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}$$

First note that

$$\binom{m}{k} = \binom{m}{m-k}$$

Thus

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} &= \sum_{k=0}^{m} \binom{m}{m-k}\binom{n}{r+k} \\
&= \sum_{j=0}^{m} \binom{m}{j}\binom{n}{r+(m-j)} \text{(re-indexing)} \\
&= \binom{m+n}{m+r} \text{(by Van der Monde)}
\end{aligned}$$

Combinatorially speaking, this identity amounts to partitioning the number of ways to choose an $(m+r)$-subset from an $(m+n)$-set, by how many elements are chosen from a specified set of m elements.

5. Extra Credit: Prove:

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$$

Proof 1 (combinatorial): The LHS counts the number of ways to first choose a committee from a group of n people, then choose a President from among the members of that committee. The RHS counts the number of ways to choose a President from among a group of n people, then choose the remainder of the committee from the remaining $n-1$ people.

Proof 2 (Binomial Theorem): A special case of the Binomial Theorem $(y = 1)$ states

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Differentiate both sides with respect to x to get:

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^k - 1$$

Plugging in $x = 1$ to the equation above gives the identity.

Proof 3 (Induction on n):
Base case: The statement is true for $n = 1$, since $\binom{1}{1} = 1 \cdot 2^0$
Inductive step: Assume true for some $m \geq 0$, i.e.

$$\binom{m}{1} + 2\binom{m}{2} + 3\binom{m}{3} + \cdots + m\binom{m}{m} = m2^{m-1}$$

We wish to prove:

$$\binom{m+1}{1} + 2\binom{m+1}{2} + 3\binom{m+1}{3} + \cdots + (m+1)\binom{m+1}{m+1} = (m+1)2^m$$

Now,

$$\binom{m+1}{1} + 2\binom{m+1}{2} + 3\binom{m+1}{3} + \cdots + (m+1)\binom{m+1}{m+1}$$

$$= (m+1)\binom{m}{0} + (m+1)\binom{m}{1} + (m+1)\binom{m}{3} + \cdots + (m+1)\binom{m}{m} \text{ (by problem 3)}$$

$$= \binom{m}{1} + 2\binom{m}{2} + 3\binom{m}{3} + \cdots + m\binom{m}{m}$$

$$+ \binom{m}{m-1} + 2\binom{m}{m-2} + 3\binom{m}{m-3} + \cdots + m\binom{m}{0}$$

$$+ \binom{m}{0} + \binom{m}{1} + \binom{m}{3} + \cdots + \binom{m}{m}$$

$$= 2^m + 2[\binom{m}{1} + 2\binom{m}{2} + 3\binom{m}{3} + \cdots + m\binom{m}{m}] \text{ (by symmetry and \# of subsets)}$$

$$= 2^m + 2 \cdot (m2^{m-1}) \text{ (by the inductive hypothesis)}$$

$$= 2^m + m2^m$$

$$= (m+1)2^m$$