Some Turán type results on the hypercube

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1 Introduction

For any graphs H, G, define $\mathrm{ex}(H, G)$ ($\mathrm{ex}^0(H, G)$) to be the maximum number of edges (vertices) which may be contained in a subgraph (induced subgraph) H' of H without H' containing G as a subgraph. The study of these quantities for various choices of H and G are known as Turán type problems. We are interested in the quantities $\mathrm{ex}(Q_n, G)$ and $\mathrm{ex}^0(Q_n, G)$, where Q_n denotes the *n-dimensional hypercube*. Q_n is the graph with $V(Q_n) = \{0,1\}^n$, and edges between vertices which differ in exactly one coordinate.

For a graph G, let $c(G, n)$ (resp. $c^0(G, n)$) be the minimum number of edges (vertices) required to intersect every copy of G in Q_n and

$$c(G) = \lim_{n \to \infty} \frac{c(G, n)}{|E(Q_n)|} = \lim_{n \to \infty} \frac{c(G, n)}{n2^{n-1}}, \qquad c^0(G) = \lim_{n \to \infty} \frac{c^0(G, n)}{|V(Q_n)|} = \lim_{n \to \infty} \frac{c^0(G, n)}{2^n}.$$

By a simple averaging argument, each ratio is non-decreasing, so the limits exist. Note $c(G, n) + \mathrm{ex}(Q_n, G) = |E(Q_n)|$, so $c(G) = 1 - \lim_{n \to \infty} \frac{\mathrm{ex}(Q_n, G)}{|E(Q_n)|}$, and $c^0(G, n) + \mathrm{ex}^0(Q_n, G) = |V(Q_n)|$, so $c^0(G) = 1 - \lim_{n \to \infty} \frac{\mathrm{ex}^0(Q_n, G)}{|V(Q_n)|}$.

The problem of determining $c(G), c(G, n), c^0(G)$, and $c^0(G, n)$ for various choices of G has been studied in extremal graph theory for many years as a variation of the original Turán problem. Alon, Krech and Szabó [1] wrote a nice introduction to these problems from this point of view. Another motivation comes from the field of parallel computers, where researchers have proposed hypercubes, and certain subgraphs of hypercubes, as architectures for parallel computation, where vertices correspond to processors and edges correspond to communication

links. Turán type problems correspond to the question of robustness, i.e. how many links (or processors) must fail before there is no copy of some desired sub-architecture. Graham, Harary, Livingston, and Stout [12] published an extensive survey of results from this perspective.

For $d \geq 2$, the only known value of $c(Q_d)$ or $c^0(Q_d)$ is $c^0(Q_2) = 1/3$, proved independently by Kostochka [18] and Johnson and Entringer [17]. Erdős [10] conjectured $\mathrm{ex}(Q_n, Q_2) = (\frac{1}{2} + o(1))|E(Q_n)|$. Currently it is known that

$$\frac{1}{2}(n + \sqrt{n})2^{n-1} \leq \mathrm{ex}(Q_n, Q_2) \lessapprox .62256|E(Q_n)|.$$

The lower bound is due to Brass, Harborth, and Neinborg [5], and the upper bound to Thomason and Wagner [20]. Bialostocki [4] proved that any subgraph of Q_n not containing Q_2 as a subgraph and intersecting every Q_2 has at most $\frac{1}{2}(n + \sqrt{n})2^{n-1}$ edges.

As one generalization of Erdős' problem, many researchers have studied $\mathrm{ex}(Q_n, C_l)$, where C_l is a cycle of length l. Chung [7] showed that if $k \geq 2, \mathrm{ex}(Q_n, C_{4k}) = o(n)2^n$, $\mathrm{ex}(Q_n, C_6) \geq \frac{1}{4}n2^{n-1}$, and $\mathrm{ex}(Q_n, C_6) \leq (\sqrt{2} - 1 + o(1))n2^{n-1}$. Conder [8] showed $\mathrm{ex}(Q_n, C_6) \geq \frac{1}{3}n2^{n-1}$ by constructing a 3-coloring of any hypercube with no monochromatic C_6. Alon, Radoičić, Sudakov, and Vondrák [2] proved for all $k \geq 5$, for all r, there exists N such that if $n > N$, every coloring of Q_n with r colors contains a monochromatic copy of C_{2k}. Furedi and Ozkahya[**?**] have shown $\mathrm{ex}(Q_n, C_{14}) = o(n)2^n$ but it is still open whether $\mathrm{ex}(Q_n, C_{10}) = o(n)2^n$. Axenovich and Martin [3] gave a 4-coloring of the edges on Q_n containing no induced copy of C_{10}.

Detjer, Emamy-K., and Guan [9], Harborth and Neinborg [14], and Graham, Harary, Livingston, and Stout [12] have studied $c(Q_d, n)$ for small values of n. Their results are listed in Section 5. It is known that $c(Q_3) \leq 1/4$, and Alon, Krech, and Szabó [1] conjectured $c(Q_3) = 1/4$. The best known lower bound was due to a result in [12] which implies $c(Q_3) \geq 1 - (5/8)^{1/4} \approx .11086$. In Section 5 we improve the lower bound for $c(Q_3)$ to $\approx .1165$. This will follow from a proof that $c(Q_3, 6) = 22$.

Alon, Krech, and Szabó [1] gave the following bounds:

$$(1 + o(1))\frac{\log d}{(d+2)2^{d+1}} \leq c(Q_d) \leq \begin{cases} \frac{4}{(d+1)^2} & \text{if } d \text{ is odd} \\ \frac{4}{d(d+2)} & \text{if } d \text{ is even.} \end{cases}$$

$$(1 + o(1))\frac{\log d}{2^{d+2}} \leq c^0(Q_d) \leq \frac{1}{d+1},$$

In Section 2, we adapt a supersaturation method of Erdős and Simonovits [11] to give some classes of graphs G for which $c(G) = c(Q_d)$ for some d.

In Section 3 we define the *Fibonacci cube of dimension d*, denoted Γ_d. The Fibonacci cube is a subgraph of the hypercube which was proposed as an architecture for parallel computing

in [16]. The fault-tolerance of Fibonacci cubes was studied by Hornh, Jiang, and Kao [15] and Caha and Gregor [6]. Gregor [13] proved $c^0(\Gamma_3) = c^0(Q_2)$. As an application of our results in Section 2 we prove that for $d \le 7, c(\Gamma_d) = c(Q_{\lceil d/2 \rceil})$ and $c^0(\Gamma_d) = c^0(Q_{\lceil d/2 \rceil})$.

Many results about Turán type problems on the hypercube come from corresponding Ramsey type questions. Call an edge coloring (vertex coloring) of a hypercube with $r \ge 2$ colors such that every copy of G contains every color, *G-polychromatic*. Denote by $p(G)$ $(p^0(G))$ the maximum number of colors with which it is possible to G-polychromatically color the edges (vertices) of *any* hypercube. Since every color class in a G-polychromatic coloring intersects every copy of G, the value of $p(G)$ gives an upper bound on $c(G)$, namely $1/p(G) \ge c(G)$. Similarly, $1/p^0(G) \ge c^0(G)$.

Alon, Krech, and Szabó [1] proved for all $d \ge 1$, $p^0(Q_d) = d + 1$ and

$$\binom{d+1}{2} \ge p(Q_d) \ge \begin{cases} \frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d+2)}{4} & \text{if } d \text{ is even,} \end{cases}$$

and subsequently it was proved in [19] that for all $d \ge 1$,

$$p(Q_d) = \begin{cases} \frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d+2)}{4} & \text{if } d \text{ is even.} \end{cases}$$

In Section 4 we give a condition which, if satisfied by a graph G, implies $p(G) \ge 3$. This implies $c(G) \le 1/3$ for these graphs.

1.1 Notation for hypercubes

We refer to the n coordinates of a vertex as *bits*, and given an edge $\{x, y\}$, we refer to the unique bit where $x_i \ne y_i$ as the *flip bit*. We represent an edge of Q_n by an n-bit vector with a star in the flip bit. For example, in Q_4, we represent the edge between vertices $[0100]$ and $[0101]$ by $[010*]$. We may denote a copy of Q_d in Q_n by an n-bit vector with stars in d coordinates. For instance $[1 * 00*]$ is the Q_2 in Q_5 with vertices $\{[10000], [11000], [10001], [11001]\}$ and edges $\{[1 * 000], [1000*], [1 * 001], [1100*]\}$. We call edges with the same flip bit *parallel*, and call the class of edges with flip bit i the ith *parallel class*. For a vertex $x \in V(Q_n)$, define the *weight* $w(x) = \sum_{i=1}^{n} x_i$, and for an edge $e \in E(Q_n)$ with flip bit j define the prefix sum $p(e) = \sum_{i=1}^{j-1} e_i$, and weight $w(e) = p(e) + \sum_{i=j+1}^{n} e_i$. The vertices (edges) of Q_n can be partitioned into *levels* where we say a vertex x (edge e) is on the ith level if $w(x) = i$ ($w(e) = i$).

2 Supersaturation results

For many graphs G, we can show that $c(G) = c(Q_d)$ or $c^0(G) = c^0(Q_d)$ for some d. The idea is to show that for fixed d and large n, if $H \subseteq Q_n$ has so many edges that it is guaranteed to contain a copy of Q_d, then it is guaranteed to contain many copies, enough so that a copy of G is also guaranteed.

2.1 Edge version

Lemma 1 *Let $\epsilon > 0$, d fixed, and let $n \to \infty$. If $H \subseteq Q_n$ has $|E(H)| \geq (1 - c(Q_d) + \epsilon)n2^{n-1}$, then there are $\Omega(n^d 2^n)$ copies of Q_d in H.*

Proof: We use a standard counting technique due to Erdős and Simonovits [11].

By decreasing ϵ, assume $0 < \epsilon < c(Q_d)$ and fix m large enough so that $c(Q_d, m) \geq (c(Q_d) - \epsilon/2)m2^{m-1}$. Suppose n is very large and we remove at most $(c(Q_d) - \epsilon)n2^{n-1}$ edges from Q_n. Then there will be some Q_d remaining in Q_n. Let λ be the proportion of copies of Q_m in Q_n which contain a copy of Q_d. We get a lower bound on λ by counting the number of edges removed in each Q_m as follows: From each of the $(1 - \lambda)\binom{n}{m}2^{n-m}$ copies of Q_m containing no copy of Q_d, at least $(c(Q_d) - \epsilon/2)m2^{m-1}$ edges must be removed, and each edge is in $\binom{n-1}{m-1}$ copies of Q_m, which gives the following inequality:

$$(1 - \lambda)\binom{n}{m}2^{n-m}(c(Q_d) - \epsilon/2)m2^{m-1} \leq \binom{n-1}{m-1}(c(Q_d) - \epsilon)n2^{n-1}.$$

This implies $\lambda \geq \frac{\epsilon/2}{c(Q_d) - \epsilon/2} > 0$, a bound independent of n.

Each Q_d is in $\binom{n-d}{m-d}$ copies of Q_m, and there are $\binom{n}{m}2^{n-m}$ copies of Q_m in Q_n, so there are at least

$$\frac{\lambda\binom{n}{m}2^{n-m}}{\binom{n-d}{m-d}} = \Omega(n^d 2^n)$$

copies of Q_d remaining in Q_n. $\square$

Theorem 2 *If T is a tree with k edges, then $c(T) = 1$.*

Proof: Suppose H is a subgraph of Q_n with $\epsilon n2^{n-1}$ edges, for some $\epsilon > 0$. Then the average degree of a vertex in H is ϵn. A theorem in graph theory states that any graph with average degree $2d$ contains a subgraph with minimum degree d, and thus contains any tree on d vertices. Thus H contains a copy of T. $\square$

The proof of Theorem 2 also implies for any $n \geq k$, $\mathrm{ex}(Q_n, T) \leq 2k2^{n-1}$, and thus $c(T, n) \geq (1 - \frac{2k}{n})n2^{n-1}$. We can generalize Theorem 2 to apply to other subgraphs of Q_n. When the context is clear, we abuse notation to define the following operations on graphs: $G_1 \cup G_2$ represents the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. $G_1 \cap G_2$ represents the graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. $G_1 \setminus G_2$ represents the graph induced on G_1 by the vertex set $V(G_1) \setminus V(G_2)$.

Define a Q_d-tree of cardinality k to be the union of k copies of Q_d, $G_1, \ldots, G_k$, such that for each $i > 1$, there is some $j < i$ s.t. $G_i \cap G_j \cong Q_{d-1}$, and $(G_i \setminus G_j) \cap \left(\cup_{l=1}^{i-1} G_l \right) = \emptyset$. For example, a tree is a Q_1-tree. Every Q_d-tree contains a copy of Q_d but contains no copy of Q_{d+1}.

Theorem 3 *If T is a Q_d-tree of cardinality k, then $c(T) = c(Q_d)$.*

Proof: $Q_d \subseteq T$, so $c(T) \leq c(Q_d)$.

For a graph $K \subseteq Q_n$, define $l(K)$ to be the dimension of the smallest subcube of Q_n containing K, and $L(K)$ to be the maximum value of $l(K')$, taken over all $K' \subseteq Q_n$ where K' is isomorphic to K. For example, if P_3 is the path with three edges and $n \geq 3$, then there are subgraphs of Q_n isomorphic to P_3 with $l = 2$ and $l = 3$, so $L(P_3) = 3$. By induction, a Q_d-tree of cardinality k has $L = d + k - 1$. Let $\epsilon > 0$ be arbitrary, n large, and H a subgraph of Q_n with $(1 - c(Q_d) + \epsilon)n2^{n-1}$ edges. We show that H contains a copy of T with l value $d + k - 1$.

Initially, let A denote the set of all copies of Q_d in H, and B be the set of all copies of Q_{d-1} in H. By Lemma 1, $|A| = \Omega(n^d 2^n)$. For $\beta \in B$, define $D(\beta) = |\{\alpha \in A : \beta \subseteq \alpha\}|$ and let $E[D]$ denote the average value of $D(\beta)$ for $\beta \in B$. There are $\binom{n}{d-1}2^{n-d+1} = O(n^{d-1}2^n)$ copies of Q_{d-1} in Q_n, so $|B| = O(n^{d-1}2^n)$. Thus $E[D] \geq an$ for some $a > 0$, independent of n. We modify A and B as follows: At each step, if there is an element $\beta \in B$ with $D(\beta) < k$, we remove it from B, and remove from A all elements which contain β. After any step, $E[D] \geq an$, since removing an element from B will cause at most k elements to be removed from A. Thus at the next step:

$$E[D] \geq \frac{\sum_{\beta \in B} D(\beta) - 2dk}{|B| - 1} \geq \frac{an|B| - 2dk}{|B| - 1} \geq \frac{an(|B| - 1)}{|B| - 1} = an.$$

When this process terminates, every element β remaining in B has $D(\beta) \geq k$.

Suppose $T = G_1 \cup \ldots \cup G_k$, as in the definition of Q_d-tree. Using induction, we may assume a copy of the Q_d-tree of cardinality $(k-1)$, $G_1 \cup \ldots \cup G_{k-1}$, is contained in H and has l value $d + k - 2$. Denote this copy by T'. Let j be an index such that $G_k \cap G_j \cong Q_{d-1}$, and $(G_k \setminus G_j) \cap \left(\cup_{l=1}^{k-1} G_l \right) = \emptyset$, and denote by G' the image of $G_k \cap G_j$ in H. Since $G' \in B$, $D(G') \geq k$. Thus the union of all members of A which contain G' has L value at least $d + k - 1$ and there is some $\alpha \in A$ such that $\alpha \cap T' = G'$. The union of α and T' is a copy of T. $\square$

Define the Cartesian product $G \times F$ of two graphs G and F to be the graph with $V(G \times F) = \{(x,y) : x \in V(G), y \in V(F)\}$ and $E(G \times F) = \{\{(x,y_0),(x,y_1)\} : x \in V(G), \{y_0,y_1\} \in E(F)\} \cup \{\{(x_0,y),(x_1,y)\} : \{x_0,x_1\} \in E(G), y \in V(F)\}$.

Theorem 4 *If $d \geq 2$, T is a Q_{d-1}-tree and F is a graph with at least one edge such that $c(F) = 1$, then $c(T \times F) = c(Q_d)$.*

Proof: $Q_d \subseteq T \times F$, so $c(T \times F) \leq c(Q_d)$.

Consider a graph $H \subseteq Q_n$ with at least $(1 - c(Q_d) + \epsilon)n2^{n-1}$ edges. Denote by e the graph with one edge. $T \times e$ is a Q_d-tree of cardinality k, so by Theorem 3, H contains some copy of $T \times e$ with $l = d + k - 1$. In fact, a supersaturation argument identical to Lemma 1 shows H contains $\Omega(n^{d+k-1}2^n)$ copies of $T \times e$ with l value $d + k - 1$.

Let S be the collection of sets of size $d+k-2$ in $[n]$. Since S has cardinality $O(n^{d+k-2})$, there is some $\sigma \in S$ and $a > 0$ such that there are at least $an2^n$ copies of $T \times e$ in H where each of its two copies of T have stars in all positions in σ. Consider the set A of all such copies of T. Since there are only a fixed finite number of copies of T in Q_{d+k-2}, some constant proportion $b > 0$ of these will have all corresponding edges parallel. Let $A' \subseteq A$ denote the set of these copies, and note $|A'| \geq abn2^n$. Construct a subgraph H' of $Q_{n-d-k+2}$ by making vertices correspond to the copies of Q_{d+k-2} with stars in the positions in σ, and putting an edge between them if the subgraph induced by the two copies of Q_{d+k-2} contains a copy of $T \times e$ which contains two elements of A'. Since H' contains $abn2^n$ edges and $c(F) = 1$, H contains a copy of F. The preimage of this graph in H is the desired copy of $T \times F$. $\square$

2.2 Vertex version

We state Lemma 5 for vertices, analogous to Lemma 1 for edges, and note that the number of copies of Q_d guaranteed in the conclusion is identical. The proof follows by the same argument. Since the other results in Section 2.1 depend only on Lemma 1, all other results are immediate. We state them here, omitting the (identical) proofs.

Lemma 5 *If $H \subseteq Q_n$ has $|V(H)| \geq (1 - c^0(Q_d) + \epsilon)2^n$, then there are $\Omega(n^d2^n)$ induced copies of Q_d in H.*

Theorem 6 *If T is a Q_d-tree of cardinality k, then $c^0(T) = c^0(Q_d)$.*

Theorem 7 *If $d \geq 2$, T is a Q_{d-1}-tree and F is a fixed graph with at least one edge such that $c(F) = 1$, then $c^0(T \times F) = c^0(Q_d)$.*

3 An application to Fibonacci cubes

The *Fibonacci cube of dimension* d, denoted Γ_d, can be defined as the subgraph of Q_d induced on vertices which do not contain 1's in consecutive coordinates. Recursively, the vertex set of Γ_d is the union of the vertex sets of Γ_{d-1} and Γ_{d-2}:

$$V(\Gamma_d) = \{u0 : u \in V(\Gamma_{d-1})\} \cup \{v01 : v \in V(\Gamma_{d-2})\} \qquad \text{for } d > 2$$

and $V(\Gamma_2) = \{[00], [01], [10]\}$, $V(\Gamma_d) = V(Q_d)$ for $d < 2$.

For all values of d, $Q_{\lceil d/2 \rceil} \subseteq \Gamma_d$ (consider the cube with 0 in every even coordinate and a star in every odd coordinate). For $d \leq 5$, Γ_d is a subgraph of a $Q_{\lceil d/2 \rceil}$-tree: $\Gamma_1 = [*]$, $\Gamma_2 = [0*] \cup [*0]$, $\Gamma_3 \subseteq [*0*] \cup [0**]$, $\Gamma_4 = [*0*0] \cup [*00*] \cup [0*0*]$, and $\Gamma_5 \subseteq [*0*0*] \cup [*00**] \cup [0*0**]$, but this is not true for $d \geq 6..$

We now have the following corollaries to the results of Section 2.

Corollary 8 *For $d \leq 7$, $c(\Gamma_d) = c(Q_{\lceil d/2 \rceil})$.*

Proof: For $d \leq 5$, $Q_{\lceil d/2 \rceil}$ is a subgraph of Γ_d, and Γ_d is a subgraph of a $Q_{\lceil d/2 \rceil}$-tree, so we may apply Theorem 3.

Let T_2 denote the tree with 2 edges with $V(T_2) = \{[00], [01], [10]\}$. For $d = 6, 7$, Γ_{d-2} is a subgraph of a $Q_{\lceil d/2 \rceil - 1}$-tree. The statement follows from Theorem 2 and Theorem 4, using the fact that $\Gamma_d \subseteq (\Gamma_{d-2} \times T_2)$. $\square$

The vertex version gives identical results, using Theorem 6 and Theorem 7.

Corollary 9 *For $d \leq 7$, $c^0(\Gamma_d) = c^0(Q_{\lceil d/2 \rceil})$.*

4 A Ramsey type theorem

Given a graph $G \subseteq Q_n$, for each vertex $v \in V(G)$ let the graph H_v have nodes $V(H_v) = \{w : vw \in E(G)\}$ and edges $E(H_v) = \{wx : v, w, \text{ and } x \text{ are in a } Q_2 \text{ in } G\}$.

Theorem 10 *If a graph G has some vertex v with H_v non-bipartite, then $p(G) \geq 3$.*

Proof: Consider the coloring $\chi : e \to 2p(e) - w(e) \pmod 3$. We will show χ is a G-polychromatic 3-coloring. Since H_v is non-bipartite, it contains an odd cycle. This corresponds to a sequence of copies of Q_2, $C_0, C_1, \ldots, C_{2k}$ such that C_i, C_{i+1} share an edge (all subscripts

are taken $\pmod{2k+1}$). We will show that the subgraph of G made up of the union of these cycles contains all three colors.

Let $m_0, \ldots, m_{2k}$ correspond to the positions of the stars in $C_0, \ldots, C_{2k}$ such that m_i, m_{i+1} are the positions of the stars in C_i. Call a sequence a, b, c *monotone* if $a < b < c$ or $a > b > c$. Consider the following two possiblilities:

(i) There is some i s.t. C_{i-1}, C_i are on the same two edge levels and m_{i-1}, m_i, m_{i+1} is monotone.

(ii) There is some i s.t. C_{i-1}, C_i are on different levels and m_{i-1}, m_i, m_{i+1} is not monotone.

If either (i) or (ii) occur, then the three edges in $C_{i-1} \cup C_i$ using the star in position m_i have all three colors. For example, assume (i) occurs, and without loss of generality $i = 1$, $m_0 < m_1 < m_2$, and the edge shared by C_0 and C_1 has 0's in positions m_0 and m_2, and color 1. Since the two squares are on the same levels the other edge in C_0 has color 0 and the other edge in C_1 has color 2. All other cases are checked similarly.

It remains to show that (i) or (ii) must occur: Suppose there is a cycle $C_0, \ldots, C_{2k}$ such that neither occurs. Let the set $S = \{C_i : m_{i-1}, m_i, m_{i+1}$ is monotone$\}$. Since there must be an even number of non-monotone triples on a cycle, $|S|$ is odd. Let $T = \{C_i : C_{i-1}, C_i$ are on different levels $\}$. T must be even. But for neither (i) nor (ii) to happen, $S = T$, which is impossible. $\square$

Thus if a graph G has some vertex v with H_v non-bipartite, then $c(G) \leq 1/3$. An example of such a graph is obtained by deleting one vertex from Q_3.

The following proposition gives a graph G where H_v is bipartite for all $v \in V(G)$, but $p(G) = 3$, which shows that the converse of Theorem 10 is not true.

Proposition 11 *Let G be the graph obtained by removing two parallel edges of Q_3 that are not incident to any common edges. Then $p(G) \geq 3$.*

Proof: It suffices to show that any embedding of G in Q_n must contain edges on three consecutive edge levels. Then if edge levels are colored $\pmod 3$, this coloring is G-polychromatic.

Consider an embedding of G and denote by C_1 and C_2 the the two copies of Q_2 in G. For $i = 1, 2$, denote by x_i and y_i the two degree three vertices at distance two in $V(C_i)$, where $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are edges in G. Since the distance from x_i to y_i is two for $i = 1, 2$, these two edges must be parallel (if they were not parallel, the distance between one of the pairs would have to be 4). Since x_i and y_i differ in the same two coordinates for $i = 1, 2$, the corresponding edges in C_1 and C_2 are parallel. Thus x_1 is on the same vertex level with respect to the three

vertex levels covered by C_1 as x_2 is with respect to those covered by C_2. Since x_1 and x_2 are on consecutive vertex levels, and C_1 and C_2 have edges on two levels each, C_1 and C_2 contain edges on three consecutive levels. $\square$

5   A new lower bound on $c(Q_3)$

In this section we prove

Theorem 12 $c(Q_3, 6) = 22$

and use it to establish a new lower bound on $c(Q_3)$.

Before we prove Theorem 12, we need a few preliminaries. Note that the union of any k parallel classes in Q_n consists of 2^{n-k} disjoint copies of Q_k. Call a set of edges intersecting every Q_d in Q_n an *(n,d)-cover*. We will call two covers *isomorphic* if and only if there is an automorphism of the cube that maps one to the other.

We will use the following observation repeatedly:

Lemma 13
$$c(Q_d, n+1) \geq \left\lceil \frac{2(n+1)c(Q_d, n)}{n} \right\rceil$$

Proof: Consider a minimum cardinality $(n+1, d)$-cover A. Without loss of generality assume the $(n+1)^{st}$ parallel class contains at least as many edges of A as any other class. Then the union of the other n parallel classes consists of two disjoint Q_n's, so they must contain at least $2c(Q_d, n)$ edges in A. The assumption of maximality on the $(n+1)^{st}$ class proves the lemma. $\square$

We will use many known results about $c(Q_d, n)$ for small values of n.

$d = 2$:   – $c(Q_2, 2) = 1$.

– $c(Q_2, 3) = 3$. (Lower bound: Lemma 13, upper bound: $\{[*00], [1*1], [01*]\}$, and the minimum cover is unique.)

– $c(Q_2, 4) = 8$. (Lower bound: Lemma 13, upper bound:$\{[*000], [*111], [1*01], [0*10], [00*1], [11*0], [101*], [010*]\}$. Dejter, Emamy-K., and Guan [9] proved that this construction is unique.)

– $c(Q_2, 5) = 24$. (see [9])

– $c(Q_2, 6) = 60$. (see [14])

$d = 3$: $-$ $c(Q_3, 3) = 1$.

  $-$ $c(Q_3, 4) = 3$. (Lower bound: Lemma 13, upper bound: $\{[*000], [0*01], [111*]\}$. The minimum (4,3)-cover is not unique, but the three edges must be in 3 different classes.)

  $-$ $c(Q_3, 5) = 8$. (Lower bound: Lemma 13, upper bound: $\{[*0000], [*1110], [1*011], [0*101], [00*11], [11*01], [101*0], [010*0]\}$. We will show that this cover is unique.)

It was previously known that $20 \le c(Q_3, 6) \le 22$ (Lower bound: Lemma 13, upper bound [12]).

We call a set of edges $A' \subseteq Q_n$ the *k-projection* of a set $A \subseteq Q_{n+1}$ if A' is obtained from A by identifying all vertices in Q_{n+1} which differ only in the kth coordinate. For instance, the minimum $(4, 2)$-cover given above is a 5-projection of the minimum $(5, 3)$-cover which was given. Note that k-projecting a set of edges corresponds to deleting the kth coordinate in each of their vector representations.

Lemma 14 *Suppose an $(n+1, d)$-cover A contains no edges in the kth parallel class. Then the k-projection A' of A is an $(n, d-1)$-cover.*

Proof: If A' is not an $(n, d-1)$-cover, then it is possible to find a $Q_{d-1} \subseteq Q_n$ containing no edges of A'. The preimage of this cube is two parallel copies of $Q_{d-1} \subseteq Q_{n+1}$ containing no edges of A and connected by edges in the kth class. Since there are no edges of A in the kth class, this is a Q_d containing no edges of A. $\square$

Lemma 15 *If A is a $(5, 3)$-cover with 8 edges, then one parallel class contains no edges in A.*

Proof: Let A be a $(5, 3)$-cover with 8 edges, and at least one edge in each class. No class contains 3 edges in A, since otherwise there would be some Q_4 covered by only 2 edges, which contradicts $c(Q_3, 4) = 3$. Thus, A contains 2 edges in 3 classes, and one edge in 2 classes. Denote the edges of A by $a = [a_1 a_2 \ldots a_5], b = [b_1 \ldots b_5], \ldots, h = [h_1 \ldots h_5]$ and without loss of generality assume $a_1 = b_1 = *, c_2 = d_2 = *, e_3 = f_3 = *, g_4 = *$, and $h = [0000*]$. Note that the matrix M whose rows are $a, \ldots, h$ must have the property that for every 2-bit string xy, every pair of columns contains some row whose entries are xy. If a matrix has this property, we say it is *2-independent*.

Suppose $g_1 = 0$. The cubes $[1*0**], [1*1**], [10***]$, and $[11***]$ do not contain edges g or h, so $c_1 = d_1 = e_1 = f_1 = 1$, since one of the edges c, d, e, f must be in each of these cubes. But then the cube $[0***\overline{g_5}]$, where $\overline{g_5} \neq g_5$ does not contain any edges, a contradiction. Thus $g_1 = 1$ and by symmetry, $g_2 = g_3 = 1$.

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$$
\begin{bmatrix}
* & 0 & 1 & a_4 & a_5 \\
* & 1 & 0 & b_4 & b_5 \\
0 & * & 1 & c_4 & c_5 \\
1 & * & 0 & d_4 & d_5 \\
0 & 1 & * & e_4 & e_5 \\
1 & 0 & * & f_4 & f_5 \\
1 & 1 & 1 & * & g_5 \\
0 & 0 & 0 & 0 & *
\end{bmatrix}
\qquad
\begin{bmatrix}
* & 0 & 0 & 0 & a_5 \\
* & 1 & 1 & 1 & b_5 \\
1 & * & 0 & 1 & c_5 \\
0 & * & 1 & 0 & d_5 \\
0 & 0 & * & 1 & e_5 \\
1 & 1 & * & 0 & f_5 \\
1 & 0 & 1 & * & g_5 \\
0 & 1 & 0 & * & h_5
\end{bmatrix}
$$

Figure 1: Left: M in Lemma 15. Right: M in Lemma 16

Since the first 3 columns of M must be 2-independent, without loss of generality we may assume $0 = a_2 = b_3 = c_1 = d_3 = e_1 = f_2$, and $1 = a_3 = b_2 = c_3 = d_1 = e_2 = f_1$ (see the first matrix in Figure 1). Now consider cubes with stars in positions 2 and 3, but not 1. Since d or f must be in the cubes $[1**1*]$, $[1**0*]$, and $[1***\overline{g_5}]$ we conclude $d_4 \neq f_4$, and at least one of d_5 and f_5 is $\overline{g_5}$. Since c or e must be in the cubes $[0***1]$, $[0***0]$, and $[0**1*]$, we conclude $c_5 \neq e_5$ and at least one of c_4 and e_4 is 1. Similarly, considering cubes with stars in positions 1 and 3 but not 2, we derive the following: $b_4 \neq e_4$, b_5 or $e_5 = 1$, $a_5 \neq f_5$, a_4 or $f_4 = 1$. Considering cubes with stars in positions 1 and 2 but not 3, we get $a_4 \neq c_4$, a_5 or $c_5 = 1$, $b_5 \neq d_5$, b_4 or $d_4 = 1$. One can verify that the above constraints imply the following: $b_4 = c_4 = f_4$, $a_4 = d_4 = e_4$, $b_5 = c_5 = f_5$, $a_5 = d_5 = e_5$. These conditions make it impossible for all four 2-bit strings to be present in the fourth and fifth columns of M, contradicting the 2-independence of M. $\square$

Lemma 16 *If A is a $(5,3)$-cover with 8 edges, then it is unique up to isomorphism.*

Proof: From Lemma 15, we know that one class, say the fifth, contains no edges in A. Again denote the 8 edges of A by $a, \ldots, h$, and the matrix where these are the rows by M. Consider the 5-projection of A. By Lemma 14 this projection must be a $(4,2)$-cover containing only 8 edges. Since the 8-edge $(4,2)$-cover is unique up to isomorphism, this means that the first four columns of M are determined up to isomorphism as well. Without loss of generality we may assign entries in M as in the second matrix in Figure 1.

It remains to check a couple of cases to see that there is only one way to assign values to the fifth column of M:

$a_5 \neq b_5$: Without loss of generality we may assume $a_5 = 0, b_5 = 1$. For columns 1 and 5 to contain all four 2-bit strings, d_5, e_5, and h_5 cannot all be the same. If two of them are 1, then

without loss of generality we may assign $d_5 = e_5 = 1, h_5 = 0$, then note that for columns 3 and 5 to contain 01, $c_5 = 1$. But then columns 4 and 5 cannot contain 10. If only one of d_5, e_5 and h_5 are 1, we may assume $h_5 = 1$, then note that for columns 4 and 5 to contain 01, $f_5 = 1$. But then columns 2 and 5 do not contain 10.

$a_5 = b_5$: Without loss of generality we may assume $a_5 = b_5 = 0$. As before, not all of d_5, e_5, and f_5 may be the same. We may check as above that it is impossible for two of them to be zero. If we set two of them equal to one, it extends to a unique assignment, and all possible choices can be mapped to each other by a cube isomorphism of Q_5. One possible representation is $(a_5, \ldots, h_5) = (0, 0, 1, 1, 1, 1, 0, 0)$. $\square$

Proof of Theorem 12: Assume for the sake of contradiction that $c(Q_3, 6) < 22$ and choose a $(6, 3)$-cover A with 21 edges.

Case 1: There is some class with 5 edges in A, say the sixth (Note no class may contain six edges since then some Q_5 would contain only 7 edges, contradicting $c(Q_3, 5) = 8$.). There can only be one such class, since if there were only 11 edges among 4 classes, in the union of these four classes, there would be some Q_4 with at most two edges, contradicting $c(Q_3, 4) = 3$. Since $\lceil (21 - 5)/5 \rceil = 4$, there is another class with 4 edges, say the fifth. Denote by W, X, Y, Z, respectively, the Q_4's $[****00], [****01], [****10], [****11]$. Each of these Q_4's must contain exactly 3 edges of A, and since $c(Q_3, 5) = 8$, at least two edges of A must go between each of the following pairs: WX, WY, XZ, and YZ. Without loss of generality, we assume there are three edges of A between W and X, and two between the other pairs. The Q_5 determined by the pair WY contains exactly 8 edges of A, and so by Lemma 15 there must be some class in WY which contains no edges of A, say the fourth. Since Y contains no edges of A in the fourth class, it must contain edges in classes 1,2 and 3. Thus YZ is a Q_5 containing 8 edges of A, some of which are in classes 1, 2, 3, and 6. By Lemma 15, we conclude that YZ contains no edges of A in the fourth class. Similarly we may consider XZ and note that it can contain no edges in the fourth class. Thus there are no edges in A the fourth class at all. If we consider the 4-projection of A, by Lemma 14 we get a $(5, 2)$-cover of cardinality 21. But this contradicts the fact that $c(Q_2, 5) = 24$.

Case 2: A has at most four edges in a class. Similar to the previous case, we may assume without loss of generality that A contains four edges each in classes 5 and 6. Two edges each connect the pairs WX, WY, XZ, and YZ, since otherwise there is a Q_5 covered by 8 edges of A, where there is a parallel class with exactly one edge, while in the unique $(5, 3)$-cover of Lemma 16, each nonempty parallel class has two edges. We may assume each of W, X, and Y contain exactly 3 edges in A, with one each in classes 1, 2, and 3. Then Z contains four edges of A.

Lemma 16 allows us to assign the edges in WX. Without loss of generality, we may assume that the following 8 edges are in A: $[1 * 1000], [01 * 100], [*00100], [0 * 0001], [*11101], [10 * 101], [11000*][00100*]$. Using Lemma 16 again it is not hard to check that the following 5 edges must be the remaining ones in WY: $[0 * 0010], [10 * 110], [*11110], [1100 * 0], [0010 * 0]$.

At most two of the edges in Z must be in class 4, since there must be edges in at least three classes in Z. But if fewer than 2 edges of Z are in class 4, then there are at most 5 remaining unassigned edges in classes 4, 5, and 6. Each can be in exactly one cube of type $[abc * **]$ where $a, b, c \in \{0, 1\}$. There are 8 such cubes in total, and only $[110 * **]$ and $[001 * **]$ contain edges already assigned in A, leaving 6 unaccounted for, a contradiction. Thus there must be exactly two edges of A in the fourth class in Z.

Of the remaining unassigned edges, one each must be contained in the Q_3's $[* * *011]$ and $[* * *111]$, and one more each in the Q_4's $[* * *01*]$ and $[* * *0 * 1]$. This leaves 2 remaining unassigned edges. For all possible assignments made thus far, there are at least 7 Q_3's of the form $[abc1 * *]$ where exactly one of a, b, c is a star which do not contain an edge in A. However each remaining unassigned edge can be in at most three such Q_3's, and thus A cannot intersect every Q_3. $\square$

We can use Theorem 12 to improve the lower bound for $c(Q_3)$ in [12] which was $1 - (5/8)^{1/4} \approx .11086$.

Theorem 17 $c(Q_3) \geq \frac{28,625}{245,760}$.

Proof: $c(Q_3, 6) = 22$ implies that $c(Q_3) \geq 22/192 \approx .11458$. We can improve this using Lemma 13, which implies

$$c(Q_3, 7) \geq \left\lceil \frac{2 \cdot 7 \cdot c(Q_3, 6)}{6} \right\rceil = \left\lceil \frac{308}{6} \right\rceil = 52,$$

giving a new bound of $c(Q_3) \geq 52/448 \approx .11607$. We can use Lemma 13 repeatedly in this way to get $c(Q_3, 8) \geq 119, c(Q_3, 9) \geq 268$, and so forth until we get $c(Q_3, 15) \geq 28,635$, at which point Lemma 13 ceases to improve the bound on $c(Q_3)$. Thus our best lower bound for $c(Q_3)$ is

$$c(Q_3) \geq \frac{28,625}{245,760} \approx .116516. \square$$

6 Open questions

There are many more questions than results for Turán type problems on Q_n. It would be interesting to find better bounds for $c(G)$ or $c^0(G)$ for any of the graphs G discussed in the paper. It would also be interesting to discover which values the functions c or c^0 may take, and

to further characterize classes of graphs which have the same value. In particular, it would be interesting to know whether $c(\Gamma_d) = c(Q_{\lceil d/2 \rceil})$ or $c^0(\Gamma_d) = c^0(Q_{\lceil d/2 \rceil})$ for values of d greater than seven.

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References

[1] N. Alon, A. Krech and T. Szabó, Turán's theorem in the hypercube, *SIAM J. on Disc. Math.* **21** (2007), 66-72.

[2] N. Alon, R. Radoičić, B. Sudakov, J. Vondrák, A Ramsey-type result for the hypercube, *J. Graph Theory* **53** (2006), 196-208.

[3] M. Axenovich, R. Martin, A note on short cycles in a hypercube, Discrete Math. 306 (2006) no. 18, 2212-2218.

[4] A. Bialostocki, Some Ramsey type results regarding the graph of the *n*-cube, *Ars Combinat.* **16-A** (1983), 39-48.

[5] P. Brass, H. Harborth, H. Neinborg, On the maximum number of edges in a C_4-free subgraph of Q_n, *Journal of Graph Theory* **19** No. 1 (1995), 17-23.

[6] R. Caha, P. Gregor, Embedding Fibonacci cubes into cubes with $\Omega(2^{cn})$ faulty vertices, in: Proceedings of the 25th MFCS'2000, Lecture Notes in Computer Science, vol. 1893, 2000, pp.253–263.

[7] F. Chung, Subgraphs of a hypercube containing no small even cycles, *Journal of Graph Theory* **16** No. 3 (1992), 273-286.

[8] M. Conder, Hexagon-free subgraphs of hypercubes, *Journal of Graph Theory* **17** No. 4 (1993), 477-479.

[9] I. J. Dejter, M. R. Emamy-K., P. Guan, On the fault tolerance in a 5-cube, *Congressus Numerantium* **80** (1991), 171-176.

[10] P. Erdős, Some Problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics*, B. Bollobás, ed., Academic Press (1984) 1-17.

[11] P. Ersős, M. Simonovits, Supersaturated graphs and hypergraphs, Combinatorica 3 (2) (1983) p. 181-192.

[12] Z. Furedi, L. Ozkayha, in preparation.

[13] N. Graham, F. Harary, M. Livingston, Q. Stout, Subcube fault-tolerance in hypercubes, *Information and Computing* **102** (1993), 280-314.

[14] P. Gregor, Recursive fault-tolerance of Fibonacci cube in hypercubes, Discrete Mathematics 306 (2006) p. 1327-1341.

[15] H. Harborth, H. Neinborg, Maximum number of edges in a six-cube without four-cycles, *Bulletin of the ICA*, Vol. 12 (1994), 55-60

[16] S.-J. Hornh, F.-S. Jiang, T.-W. Kao, Embedding of generalized Fibonacci cubes in hypercubes with faulty vertices, IEEE Trans. Parallel Distributed Systems 8 (1997) 727-737.

[17] W.-J. Hsu, Fibonacci cubes–a new interconnection topology, IEEE Trans. Parallel Distributed Systems 4 (1993) 3–12.

[18] K. A. Johnson, R. Entringer, Largest induced subgraphs of the n-cube that contain no 4-cycles, Journal of Combinatoria Theory, Series B 46 (1989) 346-355.

[19] E. A. Kostochka, Piercing the edges of the n-dimensional unit cube, (Russian) Diskret Analiz Vyp. 28 Metody Diskretnogo Analiza v Teorii Grafov i Logiceskih Funkcii (1976), 55-64.

[20] D. Offner, Polychromatic colorings of the hypercube, *SIAM J. on Disc. Math.* to appear.

[21] A. Thomason, P. Wagner, Bounding the size of square-free subgraphs of the hypercube, Discrete Mathematics, to appear.