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POLYCHROMATIC COLORINGS OF SUBCUBES OF THE HYPERCUBE*

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Abstract. Alon, Krech, and Szabó [*SIAM J. Discrete Math.*, 21 (2007), pp. 66–72] called an edge-coloring of a hypercube with p colors such that every subcube of dimension d contains every color a *d-polychromatic p-coloring*. Denote by p_d the maximum number of colors with which it is possible to d-polychromatically color *any* hypercube. We find the exact value of p_d for all values of d.

Key words. coloring of graphs and hypergraphs, extremal problems, Ramsey theory

AMS subject classifications. 05C15, 05C35, 05D10

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1. Introduction. Denote by Q_n the n-dimensional hypercube, and, for fixed graphs G, H, denote by $\mathrm{ex}(G, H)$ the maximum number of edges in a subgraph of G which does not contain a copy of H. Problems about this quantity are called Turán-type problems, after P. Turán, who gave an exact value for $\mathrm{ex}(G, H)$ when G and H are cliques. There is a rich history of studying Turán-type problems where $G = Q_n$. To cite some well-known examples, Erdős [7] conjectured that $\mathrm{ex}(Q_n, Q_2) = (\frac{1}{2} + o(1))e(Q_n)$. Currently it is known that

$$\frac{1}{2}(n + \sqrt{n})2^{n-1} \le \mathrm{ex}(Q_n, Q_2) \lessapprox .623e(Q_n).$$

The lower bound is due to Brass, Harborth, and Neinborg [4] and the upper bound to Chung [5]. Bialostocki [3] proved that any subgraph of the cube not containing Q_2 as a subgraph and intersecting every Q_2 has cardinality at most $(n + \sqrt{n})2^{n-2}$. As one generalization of Erdős' problem, Erdős [7], Chung [5], Conder [6], and Alon et al. [2] also looked at the size of $\mathrm{ex}(Q_n, C_{2k})$, where C_{2k} is a cycle of even length.

More recently, Alon, Krech, and Szabó [1] asked how large $\mathrm{ex}(Q_n, Q_d)$ is and studied related problems. In particular, they called an edge-coloring of a hypercube with p colors such that every subcube of dimension d contains every color a *d-polychromatic p-coloring*. Denote by p_d the maximum number of colors with which it is possible to d-polychromatically color *any* hypercube. (Note that the value of p_d gives a lower bound on $\mathrm{ex}(Q_n, Q_d)$; in particular, $e(Q_n)(1 - 1/p_d) \le \mathrm{ex}(Q_n, Q_d)$.) They proved the following.

THEOREM 1.

$$\binom{d+1}{2} \ge p_d \ge \begin{cases} \frac{(d+1)^2}{4} & \textit{if } d \textit{ is odd,} \\ \frac{d(d+2)}{4} & \textit{if } d \textit{ is even.} \end{cases}$$

The lower bound was obtained by construction and the upper bound by a surprising application of Ramsey's theorem, one version of which states that for all r, k there exists an n such that, if the edges of a d-uniform hypergraph on at least n vertices are

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colored with k colors, there exists a complete monochromatic subgraph on r vertices. See [8] for a full treatment.

Alon, Krech, and Szabó asked whether it was possible to resolve the behavior of p_d. We will prove that for all d, p_d is equal to the lower bound in Theorem 1, i.e., the following.

THEOREM 2.

$$p_d = \begin{cases} \frac{(d+1)^2}{4} & \textit{if } d \textit{ is odd,} \\ \frac{d(d+2)}{4} & \textit{if } d \textit{ is even.} \end{cases}$$

2. Notation. We follow the notation of [1] in denoting vertices, edges, and subcubes: A vertex of Q_n is represented by a vector of n bits. An edge corresponding to two vertices whose vectors differ in one coordinate is represented by a bit vector with a * in that coordinate. A d-dimensional subcube D is represented by a bit vector with d *'s, where the vertices of D are those vectors obtained by replacing the *'s with any combination of bits.

3. Proof of Theorem 1. For completeness, since the proof of Theorem 2 will use ideas from the proof of Theorem 1 in [1], and since the proof is very nice, we sketch it here.

Proof of Theorem 1 *(lower bound).* Assume that d is odd. The following $\frac{(d+1)^2}{4}$-coloring of Q_n is d-polychromatic: Color the edges with elements of $\mathbf{Z}_{\frac{d+1}{2}} \times \mathbf{Z}_{\frac{d+1}{2}}$ as follows: If an edge has a 1's to the left of its *, and b 1's to the right of its *, assign it the color $(a \pmod{\frac{d+1}{2}}, b \pmod{\frac{d+1}{2}})$. For any subcube Q_d, find the * with exactly $\frac{d-1}{2}$ *'s on each side of it. Note that, for this subcube, the set of edges with this * contains all colors. The construction is the same if d is even, except a is taken mod $\frac{d}{2}$ and b is taken mod $\frac{d+2}{2}$, and for any subcube Q_d we consider the edges whose star is the $\frac{d}{2}$th from the left. □

Proof of Theorem 1 *(upper bound).* Suppose that we have a d-polychromatic p-coloring c of Q_n, with n huge. We will use Ramsey's theorem for d-uniform hypergraphs with $p^{d2^{d-1}}$ colors. We define a $p^{d2^{d-1}}$-coloring of the d-subsets of $[n]$. Fix an arbitrary ordering of the edges of Q_d. For an arbitrary subset S of the indices, define $cube(S)$ to be the subcube whose * coordinates are at the positions of S and all other coordinates are 0. Let S be a d-subset of $[n]$, and define the color of S to be the vector whose coordinates are the c-values of the edges of the d-dimensional subcube $cube(S)$ (according to our fixed ordering of the edges of Q_d). By Ramsey's theorem, if n is large enough, there is a set $T \subseteq [n]$ of $d^2 + d - 1$ coordinates such that the color-vector is the same for any d-subset of T. Fix a set S of d particular coordinates from T: those which are the (id)th elements of T for $i = 1, \ldots, d$. Any two elements of S have at least $d - 1$ elements of T in between.

Call a hypercube coloring *Ramsey* if the color of an edge is determined by the number of 1's to the left of its * and the number of 1's to the right of its *.

LEMMA 3. *The coloring of* $cube(S)$ *is Ramsey.*

Proof. Let e_1 and e_2 be two edges of $cube(S)$ such that they have the same number of 1's to the left of their respective * and the same number of 1's to the right as well. Since there are at least $d - 1$ elements in T in between each coordinate of S, it is possible for us to find a set of d coordinates $S' \subseteq T$ and an edge e_3 of $cube(S')$ such that

(i) e_3 is the same edge when restricted to S as e_1 and

(ii) e_3 occupies the same position in the ordering of edges in $cube(S')$ as e_2 occupies in $cube(S)$.

Thus $c(e_1) = c(e_3) = c(e_2)$, and the lemma is proved. □

To finish the proof of the upper bound in Theorem 1, note that there are exactly $1 + 2 + \cdots + d = \binom{d+1}{2}$ ways to separate at most $d-1$ 1's by a *. By Lemma 3, a d-polychromatic edge-coloring is not possible with more colors. □

4. Proof of Theorem 2.

LEMMA 4. *If a d-polychromatic p-coloring exists for any n, then one of Ramsey-type exists.*

Proof. In the proof of the upper bound of Theorem 1, we showed that, in a sufficiently large hypercube, a Ramsey-colored d-dimensional subcube exists by applying Ramsey's theorem to a $p^{d2^{d-1}}$-coloring of the d-subsets of the indices of $[n]$. To prove that, for a sufficiently large hypercube, a subcube of any dimension k exists with a Ramsey coloring, simply apply Ramsey's theorem to a $p^{k2^{k-1}}$-coloring of the k-subsets of $[n]$, defined in an analogous way, and proceed in an identical fashion. Thus for any $p \leq p_d$, one can show the existence of a Ramsey p-colored d-polychromatic hypercube of dimension k by taking any sufficiently large p-colored d-polychromatic hypercube and applying Ramsey's theorem. □

Proof of Theorem 2. Fix d. By Lemma 4, we may consider a Ramsey d-polychromatic p_d-coloring on an arbitrarily large hypercube. Since the coloring is Ramsey, every edge belongs to some color class (a, b), where there are a 1's to the left of the * and b 1's to the right. It will be convenient to arrange the color classes in a triangular array, with $(0, 0)$ in the first row, $(0, 1)$ and $(1, 0)$ in the second row, and so forth. Define the ith *row* to be those color classes (a, b), with $a + b = i$, and the ith *diagonal* to be those classes of the form (i, j) for any j. Define an $i \times j$ *rectangle* to be a set of color classes of the following form: $\{(a + \alpha, b + \beta) : 0 \leq \alpha < j, 0 \leq \beta < i\}$, and say a rectangle R is located at the ith diagonal if i is the least diagonal that intersects R. Define a *region* to be all color classes contained in some consecutive rows and consecutive diagonals.

In this configuration, a d-cube includes the following color classes, from left to right: a $d \times 1$ rectangle, corresponding to the colors of edges using the leftmost *, followed by a $(d-1) \times 2$ rectangle, corresponding to the colors of edges using the second-leftmost *, etc., until a $1 \times d$ rectangle corresponding to the rightmost *. All of these rectangles cover colors in the same d rows, and they may overlap. They must come in order from left to right, however, since, e.g., if there are a 1's to the left of some *, there are at least a 1's to the left of any * further to the right. Call such a sequence of rectangles a *cube sequence*, and denote them (left to right) by $r_1, r_2, \ldots, r_d$. Note that any cube sequence corresponds to color classes covered by at least one cube. See the examples in Figure 1.

Choose a set of d consecutive very long rows. By our assumption that the coloring is d-polychromatic, it is not possible to find a cube sequence in these rows where every rectangle in the sequence lacks a particular color. In particular, imagine picking a color A and scanning the rows from left to right, looking for a copy of r_1 not containing A. If we find one, say, at diagonal d_1, we note that all copies of r_1 to the left of d_1 contain A. Continuing from left to right, we might then seek a copy of r_2 not containing A, and if the first one we find is at diagonal d_2, then we note that every copy of r_2 between d_1 and d_2 contains A. By continuing in this manner, we may partition the rows into at most d regions where, in the ith region, every copy of r_i contains A. By repeating this for the other colors, we note that each color partitions the rows into at most d regions, where, in the ith region, each copy of r_i contains the color. By taking the intersections of all of these regions, the rows are partitioned into at most $d \cdot p_d$ regions such that for each region, for *every* color A, there is a rectangle

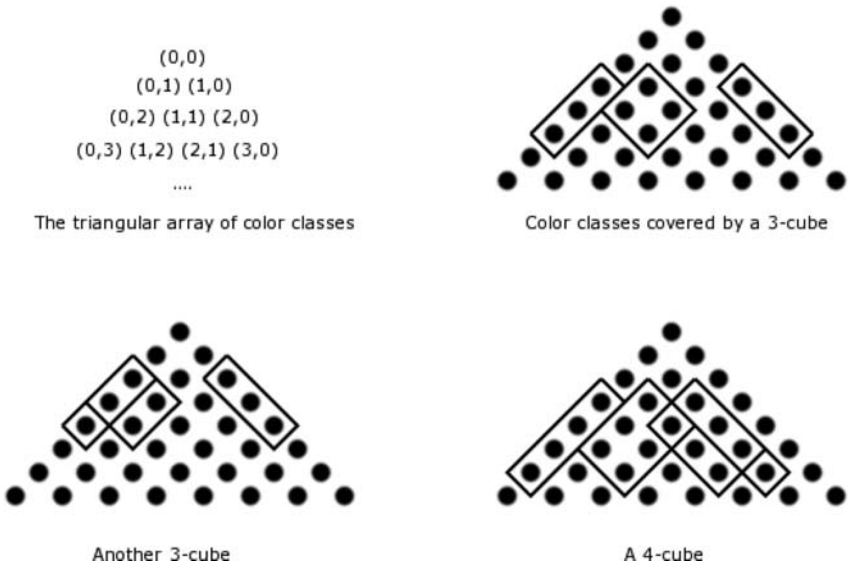

FIG. 1. *The triangular array of color classes and some cube sequences.*

r_A such that every copy of r_A in the region contains A. In particular, since we may choose the rows as long as we like, we can find an arbitrarily long region with this property.

Now consider such a region with m diagonals, where $m \gg d$. For each color A, assign d variables, $\{A_i\}_{i=1}^{d}$ corresponding to the number of times A appears in the ith row in our region. Note that, if A appears in the top or bottom row, it can be contained in at most one copy of r_A. If it appears in the second or the $d-1$st row, it can be contained in at most two copies of r_A, and so forth. There are at least $m-d$ copies of r_A in the region, and thus

$$A_1 + 2A_2 + 3A_3 + \cdots + \left\lceil \frac{d}{2} \right\rceil A_{\lceil \frac{d}{2} \rceil} + \cdots + 2A_{d-1} + A_d \geq m - d.$$

Since $A_i + B_i + \cdots = m$, if we add up these equations for each color, we get

$$m + 2m + 3m + \cdots + \left\lceil \frac{d}{2} \right\rceil m + \cdots + 2m + m \geq p_d(m-d).$$

The left-hand side is equal to $\frac{(d+1)^2}{4}m$ if d is odd and $\frac{d(d+2)}{4}m$ if d is even. Divide both sides by m, and note that $(m-d)/m$ can be as close to 1 as desired to obtain the result. $\square$

This method may be used to immediately give other similar results. For instance, say two edges in a subcube belong to the same *layer* if their vector representation contains the same number of 1's. Then, if $q_{d,k}$ $(k \leq d)$ is the maximum number such that any Q_n can be edge-colored so that the kth layer of any sub-Q_d contains all colors, then $q_{d,\lceil d/2 \rceil} = \lceil d/2 \rceil$ (e.g., $q_{3,2} = 2$).

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