Invariants and Algorithms

November 9, 2015

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1 Introduction

An *invariant* is something that doesn't change after some process. A *monovariant* is something that changes in one direction after a process, for instance something that increases.

2 Examples

1. (USAMO 1999) Let $a_1, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + \cdots + a_n \geq n \text{ and } a_1^2 + \cdots + a_n^2 \geq n^2$$

Prove that $\max\{a_1, \ldots, a_n\} \geq 2$.

Solution: Assume $a_1, \ldots, a_n < 2$. We use an algorithm that keeps $a_1 + \cdots + a_n =: s$ invariant and strictly increases $a_1^2 + \cdots + a_n^2 =: t$. For $i = 1, 2, \ldots, n-1$, replace a_i, a_{i+1} with $2, a_i + a_{i+1} - 2$. Clearly s is invariant, and t gains a total value of

$$2^2 + (a_i + a_{i+1} - 2)^2 - a_i^2 - a_{i+1}^2 = 8 + 2a_i a_{i+1} - 4a_i - 4a_{i+1} = 2(a_i - 2)(a_{i+1} - 2) > 0$$

since $a_i, a_{i+1} < 2$. One small technicality to check is that after this swap, the new value of a_{i+1} is also less than 2 (else the inequality won't hold). But that's okay, because $a_i + a_{i+2} - 2 < 2 + 2 - 2 = 2$. So at the end of all this swapping we get $2, \ldots, 2, s - 2(n-1)$. In particular, $s - 2(n-1) \geq -n + 2$ by assumption. Therefore,

$$n^2 \leq a_1^2 + \cdots + a_n^2 < 2^2 + \cdots + 2^2 + (s - 2(n-1))^2 \leq 4(n-1) + (-n+2)^2 = n^2$$

contradiction. $\qquad\square$

2. (USAMO 2003) At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Solution: Let $(a_1, \ldots, a_6)$ be the numbers on the hexagon at a given time. All numbers henceforth will be taken modulo 2. First, show any hexagon with odd sum can turn into $(1, 0, 0, 0, 0, 0)$. We have four cases: $(1, 0, 1, 0, 1, 0) \to (1, 0, 0, 0, 0, 0)$; $(1, 1, 1, 0, 1, 1) \to (1, 0, 1, 0, 1, 0)$; $(1, 1, 0, 0, 0, 1) \to (0, 1, 0, 0, 0, 1) \to (0, 0, 0, 0, 0, 1)$; $(1, 1, 0, 1, 0, 0) \to (1, 1, 0, 0, 0, 1)$. So we can assume a_1 is odd and $a_2, \ldots, a_6$ are even.

Now we show such a hexagon can be turned into another hexagon of odd sum with a smaller maximum (or all zeros). That is, $\max\{a_1, \ldots, a_6\}$ will be a decreasing monovariant. We need to do casework on where the maximum is. If it is even, then take $(1, 0, 0, 0, 0, 0) \to (1, 0, 0, 0, 0, 1) \to (1, 0, 0, 0, 1, 1) \to (1, 0, 0, 1, 1, 1) \to (1, 0, 1, 1, 1, 1) \to (1, 0, 1, 1, 1, 1)$. If it is odd, then take $(1, 0, 0, 0, 0, 0) \to (1, 0, 0, 0, 0, 1) \to (1, 1, 0, 0, 0, 1) \to (0, 1, 0, 0, 0, 1) \to (0, 0, 0, 0, 0, 1)$. It's easy to check that it was decreased in both cases. Eventually the maximum will hit 0. $\qquad\square$

3. (IMOSL 1994) Peter has three bank accounts, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can Peter always transfer all his money into one account?

Solution: The second part is easier: if the accounts have an odd sum, then the final account cannot be doubled, so the answer is no. For the first part: let the accounts have $0 < A \le B \le C$ dollars each. We prove B can be reduced to $B \pmod{A}$. Let $B = mA + n$, and consider the binary representation of $m = \overline{m_k \ldots m_1 m_0}_2$. Consider the algorithm where at the ith step, for $0 \le i \le k$, take from B if $m_i = 1$ or from C if $m_i = 0$. It works because at the beginning of the ith step, there are $2^i A$ dollars in the first account. Thus, after the ith step, we subtract $2^i m_i A$ from B. After the $k+1$ steps, the second account has $B - (2^0 m_0 + 2^1 m_1 + \cdots + 2^k m_k) = B - mA = n$. Since $n \equiv B \pmod{A}$, we are done. $\qquad\square$

4. (China 2010) There are some finite number of cards placed at the points $A_1, A_2, \ldots, A_n$ and O, where $n \ge 3$. We can perform one of the following operations in each step:

   (a) If there are more than 2 cards at some point A_i, we can remove 3 cards from this point and place one each at A_{i-1}, A_{i+1}, and O. Note that indices are taken modulo n.

   (b) If there are at least n cards at O, we can remove n cards from O and place one each at $A_1, A_2, \ldots, A_n$.

   Show that the if total number of cards is at least $n^2 + 3n + 1$, we can make the number of cards at each vertex at least $n + 1$ after finitely many steps.

   Solution: Apply the first operation until each vertex has at most 2 cards. The center must have at least $n^2 + 3n + 1 - 2n = n^2 + n + 1$ cards in it. Apply the second operation so that each vertex has 1, 2, or 3 cards. Then let a be the number of vertices with 1 card, b with 2, and c with 3. We have $a + b + c = n$, so the center must have at least

   $$n^2 + 3n + 1 - (a + 2b + 3c) = n^2 + 3n + 1 - (2n - a + c) = n^2 + n^1 + a - c$$

   Once we apply the second operation n times, each vertex will have at least $n + 1$ cards, and the center will have $n^2 + n + 1 + a - c - n^2 = n + 1 + a - c$ cards. It suffices to prove that $a \ge c$... or force the inequality to be true! One way to do this is by showing that between any two 3s, there is at least one 1, but there is a problem when the 3s are consecutive. Turn any string $x, 3, 3, \ldots, 3, y$ $(x, y \ne 3)$ into $x + 1, 1, 2, 2, \ldots, 2, 1, y + 1$. This removes consecutive 3s. Turn all $x, 3, 2, 2, \ldots, 2, 3, y$ into $x + 1, 1, 1, \ldots, 1, y + 1$. Noting that this doesn't produce consecutive 3s, we have $a \ge c$. $\qquad\square$

3   Problems

1. (UK 2011) Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

   (a) Remove an equal number of balls from each bag

   (b) Double the number of balls in one bag.

   Is it always possible to empty both bags after a finite sequence of operations? Operation 2 is now replaced with "Triple the number of balls in one bag." Is it now always possible to empty both bags after a finite sequence of operations?

2. (CGMO 2011) There are n boxes $B_1, B_2, \ldots, B_n$ from left to right, and there are n balls in these boxes.

   (a) If there is at least 1 ball in B_1, we can move one to B_2.

   (b) If there is at least 1 ball in B_n, we can move one to B_{n-1}.

   (c) If there are at least 2 balls in B_k, $2 \le k \le n - 1$ we can move one to B_{k-1}, and one to B_{k+1}.

   Prove that, for any arrangement of the n balls, we can achieve that each box has one ball in it.

3. (Putnam 2010) There are 2010 boxes labeled $B_1, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from B_i into any other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

4. (IMOSL 1989) A natural number is written in each square of an $m \times n$ chess board. The allowed move is to add an integer k to each of two adjacent numbers in such a way that non-negative numbers are obtained. (Two squares are adjacent if they have a common side.) Find a necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

5. (USAMO 2015) Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i, k), (i, l), (j, k), (j, l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively,j or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively. Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

6. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

7. (MEMO 2008) On a blackboard there are $n \geq 2, n \in \mathbb{Z}^+$ numbers. At each step, we select two numbers from the blackboard and replace both of them by their sum. Determine all numbers n for which it is always possible to yield n identical numbers after a finite number of steps.

8. (IMOSL 2013) A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

   (a) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

   (b) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

   Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

9. (IMO 2010) Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

   (a) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

   (b) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

   Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

4   Solutions

1. (UK 2011) Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

   (a) Remove an equal number of balls from each bag

   (b) Double the number of balls in one bag.

Is it always possible to empty both bags after a finite sequence of operations? Operation 2 is now replaced with "Triple the number of balls in one bag." Is it now always possible to empty both bags after a finite sequence of operations?

Solution: If $m = n$ remove all balls in the bag. WLOG $m > n$. Remove $n - 1$ balls from each bag. Iteratively double the bag with 1 ball in it to get 2 balls, and remove 1 ball from each bag. Repeat until both bags have 1 ball and finish. For the second part, note that these moves preserve $m + n \pmod 2$, so if we have $m = 2, n = 1$ for instance, we can never get $m = n = 0$. $\qquad\square$

2. (CGMO 2011) There are n boxes $B_1, B_2, \ldots, B_n$ from left to right, and there are n balls in these boxes.

   (a) If there is at least 1 ball in B_1, we can move one to B_2.

   (b) If there is at least 1 ball in B_n, we can move one to B_{n-1}.

   (c) If there are at least 2 balls in B_k, $2 \le k \le n - 1$ we can move one to B_{k-1}, and one to B_{k+1}.

   Prove that, for any arrangement of the n balls, we can achieve that each box has one ball in it.

   Solution: We use the following algorithm:

   (a) Perform the second and third operation until they can no longer be performed. This leaves boxes $B_2, B_3, \ldots, B_{n-1}$ with at most 1 ball and B_n with at most 0 balls.

   (b) Consider the smallest k such that B_k has 0 balls. That is, $(B_1, B_2, B_3, \ldots, B_{k-1}, B_k) = (x, 1, 1, \ldots, 1, 0)$.

   (c) Apply the first operation to get $(x - 1, 2, 1, \ldots, 1, 0)$.

   (d) Apply the third operation on $B_2, B_3, \ldots, B_{k-1}$ to end up with $(x, 1, 1, \ldots, 1, 1, 0, 1)$.

   (e) Repeat (b) and (c) enough to get $(x, 1, 1, \ldots, 1, 1, 0) \to (x, 1, 1, \ldots, 1, 0, 1) \to (x, 1, 1, \ldots, 0, 1, 1) \to \ldots (x, 0, 1, 1, \ldots, 1, 1)$.

   (f) Apply the first step to get $(x - 1, 1, 1, 1, 1, \ldots, 1, 1)$.

   (g) Apply (b) through (f) to get the desired result.

   The only issue is proving that (a) always terminates. To do this, weight the boxes to create a monovariant, then show that the weighted sum is bounded above. In fact, any application of the second or third operation strictly increases the sum

   $$B_0 2^n + B_1 2^{n-1} + B_2 2^{n-2} + \cdots + B_{n-1} 2^1 + B_n 2^0$$

   because

   $$2(B_{n-1} + 1) + (B_n - 1) = 2B_{n-1} + B_n + 1 > 2B_{n-1} + B_n$$

   and

   $$
   \begin{aligned}
   2^{n-i+1}(B_{i-1} + 1) + 2^{n-i}(B_i - 2) + 2^{n-i-1}(B_{i+1} + 1) &= B_{i-1} 2^{n-i+1} \\
   &\quad + B_i 2^{n-i} + B_{i+1} 2^{n-i-1} + 2^{n-i+1} - 2 \cdot 2^{n-i} + 2^{n-i-1} \\
   &> B_{i-1} 2^{n-i+1} + B_i 2^{n-i} + B_{i+1} 2^{n-i-1}
   \end{aligned}
   $$

   from $2^{n-i+1} - 2 \cdot 2^{n-i} + 2^{n-i-1} = 2^{n-i-1} > 0$. Finally, since

   $$B_0 2^n + B_1 2^{n-1} + B_2 2^{n-2} + \cdots + B_{n-1} 2^1 + B_n 2^0 \le n \cdot 2^n$$

   this algorithm must terminate, so we are done. $\qquad\square$

3. (Putnam 2010) There are 2010 boxes labeled $B_1, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from B_i into any other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

   Solution: This happens precisely when $n \geq 1005$. When $n \leq 1005$, we can find a construction of the sort $B_1 \leq 0, B_2 \leq 1, \ldots, B_{2010} \leq 2009$. Otherwise, perform the first operation on B_i for $i > 1$ until it can't be performed. There will be at least $2010n - (1 + 2 + \cdots + 2009) = 2010(n - 1004.5) \geq 1005$ balls in B_1. Then if any B_i has $> n$ but $< i$ balls, add enough from B_1 to make it have i balls, bring those back to B_1, and add n balls from B_1 to B_i. We can continue this until all boxes have n balls in them. $\qquad\square$

4. (IMOSL 1989) A natural number is written in each square of an $m \times n$ chess board. The allowed move is to add an integer k to each of two adjacent numbers in such a way that non-negative numbers are obtained. (Two squares are adjacent if they have a common side.) Find a necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

   Solution: Apply the chess board coloring. Let b and w be the sums in the black and white squares. We see that $b - w$ is invariant, and at the end $b - w = 0$, so $b = w$ is necessary.

   Now we prove it is sufficient. Consider the numbers in squares (x, y, z) adjacent in that order, and we want to find an algorithm to make $x = 0$. If $x \leq y$, apply $(x, y, z) \to (0, y - x, z)$. If $x \geq y$, apply $(x, y, z) \to (x, x, z + x - y) \to (0, 0, z + x - y)$. We can apply this algorithm first to all the rows from left to right, then to all the columns from top to bottom to reduce the problem to a 2×2 square.

   In the 2×2 square, we can make at least one number in each row 0. If they are in the same column, then $b = w$ implies that the remaining numbers are equal, so eliminate them. If they are in opposite columns, then one of b, w is 0, so all entries must be 0. $\qquad\square$

5. (USAMO 2015) Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i, k), (i, l), (j, k), (j, l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively,j or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively. Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

   Solution: The invariant is the column and row sums. To prove that row and column sums form equivalence classes, consider two boards B, C with the same row/column sums, and let $A = B - C$ be the term-wise difference of the number of stones on each square of the grid. Note that entries of A can be negative. At the leftmost column, starting from the top, perform some stone moves until you get a full column of 0s in A. Then do that for second leftmost column, etc., to get only 0s on the board of A. The number of possible column sums and row sums, using the stars and bars method, is $\binom{m+n-1}{m}^2$. $\qquad\square$

6. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

   Solution: Consider the product of the number on all of the sheets. Since $(x + y)^2 \geq 4xy$, each step increases the product by at least a factor of 4. Thus, at the end, the product is at least $4^{m \cdot 2^{m-1}}$. By AM-GM, the sum is at least $2^m \sqrt[2^m]{4^{m \cdot 2^{m-1}}} = 4^m$. $\qquad\square$

7. (MEMO 2008) On a blackboard there are $n \geq 2, n \in \mathbb{Z}^+$ numbers. At each step, we select two numbers from the blackboard and replace both of them by their sum. Determine all numbers n for which it is always possible to yield n identical numbers after a finite number of steps.

   Solution: For odd n, we quickly notice the counterexample $(2, 2, \ldots, 2, 1)$. Let m be the maximum among the numbers. The invariant here is the parity of the number of terms that are equal to m, which must end up even.

We assert that all even numbers work. Let $a_1, a_2, \ldots, a_{2n}$ be the numbers on the board. Since n is even, we can group the terms into pairs: $(a_1, a_2), (a_3, a_4), \ldots, (a_{2n-1}, a_{2n})$. Furthermore, let $b_1, b_2, \ldots, b_{2n}$ be $a_1, a_2, \ldots, a_{2n}$ divided by the largest power of 2 that divides all terms. Then, perform the following algorithm:

(a) For each $1 \leq i \leq n$, replace (a_{2i-1}, a_{2i}) with $(a_{2i-1} + a_{2i}, a_{2i-1} + a_{2i})$. Now $a_{2i-1} = a_{2i}$ for $1 \leq i \leq n$.

(b) For each $1 \leq i \leq n$, if $a_{2i-1} = a_{2i}$ is odd, replace (a_{2i}, a_{2i}) with $(2a_{2i}, 2a_{2i})$. Repeat this step until each of $a_1, a_2, \ldots, a_{2n}$ has the same common power of 2, i.e., $b_1, b_2, \ldots, b_n$ are all odd. This step can be referred to as "dividing" a pair (b_{2i-1}, b_{2i}) by 2.

(c) Add the largest pair with the smallest pair.

(d) Return to step 2.

This algorithm may look somewhat complex, but it is actually fairly easy:

a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	
3	4	4	2	3	3	3	4	4	2	3	3	initial
7	7	6	6	6	6	7	7	6	6	6	6	(1)
14	14	6	6	6	6	7	7	3	3	3	3	(2)
20	20	20	20	6	6	10	10	10	10	3	3	(3)
20	20	20	20	12	12	5	5	5	5	3	3	(2)
32	32	20	20	32	32	8	8	5	5	8	8	(3)
32	32	160	160	32	32	1	1	5	5	1	1	(2)
192	192	192	192	32	32	6	6	6	6	1	1	(3)
192	192	192	192	256	256	3	3	3	3	1	1	(2)
256	256	192	192	256	256	4	4	3	3	4	4	(3)
256	256	768	768	256	256	1	1	3	3	1	1	(2)
1024	1024	1024	1024	256	256	4	4	4	4	1	1	(3)
1024	1024	1024	1024	1024	1024	1	1	1	1	1	1	(2)

Now we have to prove that this algorithm terminates. Consider a set of $(b_1, b_2, \ldots, b_n)$ with all odd elements and $b_1 + b_2 + \cdots + b_n > 6$ (else we are already done). It's clear that we can such a state from any given position. After applying (3) and (2), we assert that $b_1 + b_2 + \cdots + b_n$ has decreased. Without loss of generality, let (b_1, b_2) be the maximum pair and (b_3, b_4) the minimum pair. After the operation, we have $b'_1, b'_2, \ldots, b'_n$. We know that for some k, since $b_1 + b_3$ is even,

$$b'_1 + b'_3 = b'_1 + b'_3 = \frac{b_1 + b_3}{2^k} \leq b_1 + b_3$$

and similarly with $b_2 + b_4$. Therefore, the sum always decreases, so it will always decrease to 6. $\quad\square$

8. (IMOSL 2013) A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(a) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(b) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

Solution: Let the imons be the vertices of a graph, and connect two vertices of the graph G with an edge if the two are entagled. We assert that we can always decrease the chromatic number $\chi(G)$ (the minimum number of colors to color the vertices of a graph such that no two adjacent vertices are the same color) with an algorithm:

(a) Apply the first operation until all vertices have an even degree to get G'. Clearly $\chi(G') \leq \chi(G)$.

(b) Apply the second operation to get G''. We can still color this graph with $\chi(G')$ colors: each instance a vertex on G' was colored with c_i (out of the $\chi(G')$ colors $c_1, c_2, \ldots, c_{\chi(G')}$), we can color the copied vertex with c_{i+1}, for example. The degree of each vertex of G'' is thus odd.

(c) Delete the vertices of some color c one by one. Since no two vertices of c are connected, the degrees don't change as we perform this. The resulting graph has $\chi(G'') = \chi(G') - 1 \leq \chi(G) - 1$. $\square$

9. (IMO 2010) Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

(a) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

(b) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Solution: Let $N = 2010^{2010^{2010}}$. Let the boxes have $(a_1, a_2, \ldots, a_6)$ coins at a given moment. We assert that $(1, 1, 1, 1, 1, 1)$ can be turned into $(0, 0, 0, 0, 0, N)$. To do this, we shall prove two statements first.

We assert that $(x, 0, 0)$ can go to $(x - y, 2^y, 0)$. Indeed, $(x, 0, 0) \to (x - 1, 2^1, 0)$ and

$$(x - y, 2^y, 0) \to (x - y, 2^y - 1, 2) \to (x - y, 2^y - 1, 4) \ldots (x - y, 0, 2^{y+1}) \to (x - y - 1, 2^{y+1}, 0)$$

In particular, $(x, 0, 0)$ can go to $(0, 2^x, 0)$.

Now let $T_n = 2^{2^{2^{\cdots}}}\underbrace{\phantom{2^{2^2}}}_{n}$. We assert that $(x - y, 0, 0, 0)$ can go to $(0, T_y, 0, 0)$. Indeed, $(x, 0, 0, 0) \to (x - 1, 2, 0, 0)$ and

$$(x - y, T_y, 0, 0) \to (x - y, 0, 2^{T_y} = T_{y+1}, 0) \to (x - y - 1, T_{y+1}, 0, 0)$$

In particular, $(x, 0, 0, 0)$ can go to $(0, T_x, 0, 0)$.

To solve the original problem, consider

$$(1, 1, 1, 1, 1, 1) \to (1, 1, 1, 1, 0, 3) \to (1, 1, 1, 0, 3, 0) \to (1, 1, 0, 3, 0, 0) \to (1, 0, 3, 0, 0, 0)$$
$$\to (0, 3, 0, 0, 0, 0) \to (0, 0, 16, 0, 0, 0) \to (0, 0, 0, T_{16}, 0, 0)$$

Notice that

$$2010^{2010^{2010}} < (2^{11})^{2010^{2010}} = 2^{11 \cdot 2010^{2010}} < 2^{2010^{2011}} < 2^{(2^{11})^{2011}} = 2^{2^{11 \cdot 2011}} < 2^{2^{2^{15}}} < T_{16}$$

so apply the second operation on the last 3 boxes to reach $(0, 0, 0, N/4, 0, 0)$. Finally, apply the first operation sufficiently to reach

$$(0, 0, 0, N/4, 0, 0) \to (0, 0, 0, 0, N/2, 0) \to (0, 0, 0, 0, 0, N)$$

$\square$