21-127 Concepts of Mathematics
Carnegie Mellon University
Summer II 2012
July 2 - August 10

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Website: http://math.cmu.edu/~bwsulliv/teaching/concepts/main.html
Texts: free materials will be provided by instructor

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1. Course Summary and Content:

Outline and goals: This course will introduce you to the work of mathematicians by investigating the concept of a mathematical *proof*. How do we *know* certain facts in the mathematical universe? How can we *show* to others, in a precise and concise way, that a certain proposed statement is true, or false? How can we use our ingenuity to come up with such proposed statements in the first place? Despite popular opinion, mathematics is a truly creative and expository art, while it still retains the benefits of a rigorous, logical science. In general, we may guess at truths and verify them via experiment, but, in mathematics, the ultimate goal is to construct a proper, understandable, and irrefutable argument. The major goals of this course, therefore, are to develop (1) an appreciation for mathematical proofs, (2) the ability to construct proofs, and (3) the ability to read and critique the proofs of others. In addressing these goals, we will investigate some interesting branches of mathematics and develop some fundamental knowledge that will be useful in any further mathematical studies.

Weeks 1-3: We will start by exploring some common mathematical knowledge and breaking down how we "know" such things. We will study some famous theorems and solve some intriguing puzzles, and use these to motivate a rigorous study of what it means to be a *proof*. This will necessitate a discussion of **sets**, the fundamental objects of mathematics. We will see many examples and explore their properties and develop some standard notation. From there, we will further rigorize the idea of **logic**; we will properly explain what we mean by "and" and "or" and, what statements like "If ... then ..." actually mean, mathematically. With these tools in hand, we will develop the basic **proof techniques**, to be utilized throughout the course. Among those techniques is the widely-used idea of mathematical **induction**.

Weeks 4-6: The rest of the course will be an exploration of various related topics. Throughout, we will be applying the proof techniques and logical thinking we developed earlier, giving us practice with constructing and understanding written proofs. The first topic we will investigate is the notion of a **function**. This will first require a discussion of **binary relations**, a notion in set theory that is actually the foundation of functions, but something you've likely never considered before. During this part, we will make a brief foray into **number theory** and study some of the properties of the integers. We will then see several properties of functions and their usefulness. Specifically, we will see how functions can be used to describe the **cardinality**, or size, of a set. We will characterize finite and infinite sets, and see some rather surprising results about infinite sets and how "big" they are! Thereafter, we will restrict our attention to finite sets, and develop several techniques and theorems with the goal of being able to "count" the elements of finite sets. This branch of mathematics is known as **combinatorics**, and is a very popular and applicable area of study. Finally, we will apply the ideas developed in combinatorics to the branch of **probability**, and mathematically describe some ideas that might have only been intuitions to you, thus far. With any remaining time, we will mention some related areas of mathematics, particularly **graph theory** and other discrete structures.

2. Prerequisites and Objectives:

Assumed knowledge: As an introductory course, Concepts has *no* official prerequisites. In particular, we will not assume *any* familiarity with calculus, and we likely won't even have cause to discuss it. However, a good understanding of and comfort with high-school level algebra is assumed.

Skills to be gained: The three major goals of the course were listed above under "Outline and goals". In the process of addressing those goals, we will all be developing the ability to ...

- Read, understand and analyze statements made in mathematical notation and syntax
- Write down one's own thoughts in mathematical symbols
- Talk about mathematical ideas verbally with our peers and illustrate via examples
- Understand written proofs, like one might find in a textbook or scholarly paper
- Critique flawed proofs and identify errors of correctness, grammar, and readability
- Identify and properly utilize some standard proof techniques: existence, uniqueness, universality, direct, contrapositive, contradiction, mathematical induction, and counting in two ways
- Approach unsolved problems and identify which techniques to apply
- Summarize and prove particular famous theorems in different areas of mathematics

among other skills, of course! You will notice your own progress on these objectives as the course goes on, and the assessments (see below) are designed to reflect your progress for me to observe.

3. Schedule and Materials:

Calendar: There is a detailed calendar on the course website that contains a list of topics for each day. I intend to adhere as closely as possible to this schedule, with some inevitable small variations. Each day, I will share some summarized lecture notes, both in class and on the course website. These notes will contain important definitions and results, but will also contain some blank sections for you to fill in certain examples and proofs. As such, these are *not* meant to be a full replacement for class attendance! Rather, they are meant to help you with annotating and provide a general outline for your notes.

Texts and notes: There is no required textbook to purchase for this course. Instead, I will be sharing large portions of a project I have been working on specifically for this course, which constitutes my doctoral thesis. I have listed some other available references–both for free and for purchase–on the course website, and I encourage you to check those out if you are looking for other materials to have as references, or for another point of view. However, you should receive everything you *need* for this course from me on the website!

I will be offering a modicum of extra credit to anyone who points out errors in my materials (both book and lecture notes) or makes helpful suggestions. Please talk to me in person or send me an email if you spot anything that looks wrong or have some general comments for me!

4. Assessment and Assistance:

Your final grade will be based on two components–**homework** and **exams**– each weighted at **50%**.

Homework: There will be eight homework assignments, due mostly on Tuesdays and Fridays at the *beginning of class*. The exceptions are Tuesdays the day after a Monday exam (no homework due those days), and the final week (during which the homework is due on Wednesday). You can see these due dates marked on the course calendar. When I assign a homework, it will become available as a link on the calendar. Not all assignments will be weighted equally; rather, individual questions will be weighted according to difficulty and importance. I can guarantee, though, that no homework will be worth more than 1.5 times any other

homework. (That is, if the "smallest" homework is worth 40 points, say, then all homeworks will be worth between 40 and 60 points.) Since the sum total represents half your grade, I encourage you to be dutiful about working on your homeworks, starting them immediately when they are assigned, and asking me questions about them. Homeworks will be returned within two days of their submission. Solutions will be provided on the day of submission, so *late homework will not be accepted*. I understand that life emergencies that occur from time to time; if such a situation arises, contact me as soon as possible to discuss specific issues about turning in homework and/or attendance.

Collaboration policy: You are allowed and *encouraged* to work together on homeworks. By this, I mean that you should feel free to talk about the problems with your classmates, discuss possible strategies, and even work through a solution. However, you should have no *written record* of such collaboration, any solution you submit should be written entirely from your own mind, and you should write the names of anyone you collaborated with on the top of your assignment. For instance, you may work with a classmate on a chalkboard, but you should erase the board completely and go your separate ways before attempting to write up a solution to submit, in addition to noting who you worked with. I have lots of experience with reading submitted proofs, so *trust me* when I say that I can tell when you are not following this policy. If you have any questions about this, please talk to me in person. It is better to address these issues before they lead to any negative consequences. Ultimately, this policy is in place to *encourage* collaboration and discussion but to *discourage* any students from relying too heavily on the assistance of classmates.

Exams: There will be three exams: on the Monday of the third and fifth weeks, and on the Friday of the sixth (and final) week. You will be given the entire 80-minute lecture period to complete them. You will not be allowed to use any notes or references during the exam period. There will be three sections on the exams:

1. *Short Answer*. These questions will be designed to test your general knowledge and understanding of the topics. For example, you might have to provide a definition of a term, or state a particular theorem, or explain whether a statement is true or false.
   This section will be roughly 25% of an exam's points.

2. *Prepared Questions*. These questions will be provided to you in advance. One week before each exam, I will share a document with roughly 10 problems. You will be able to work on them with your classmates (but not me!) and develop solutions before the exam. A few of the questions (about 3-5, depending on difficulty) will then appear on the exam. The problems will represent a survey of the content, as well; working on them should be a helpful (though not exhaustive) study technique.
   This section will be roughly 50% of an exam's points.

3. *Exercises*. These questions will expect you to apply any relevant knowledge and techniques to solve problems and provide proofs for propositions you haven't yet seen. They will be closely related to homework problems, in both content and difficulty. Having a solid understanding of the material will help you in this section, even more so than the others. Also, doing well in the Prepared Questions section should give you more time to work on these problems.
   This section will be roughly 25% of an exam's points.

Your best exam of the three will be weighted at 20% of your total grade, and your other two exams will be weighted at 15%.

Extra credit: There will be an opportunity to gain extra credit points by completing a research project in mathematics. The individual details of the project should be worked out with me, in person, so you must approach me in person or via email before the end of the fourth week of the course if you wish to pursue such a project. Ideally, you will identify some interesting area of mathematics or some interesting person that you want to investigate, find some resources (both online and in print), and write 2-4 pages on the topic. This is not necessarily meant to constitute *original* research, but rather an exposition about something you find personally interesting. A submitted project can earn you up to 3% extra points on your final grade.

Letter grades: Grades will be distributed on the standard decile scale: A=90-100%, B=80-89%, C=70-79%, D=60-69%, R=0-59%. I reserve the right to lower the requirement for a certain letter, but I will

certainly not raise the bounds.

Extra help: In addition to time spent in class, you will have the opportunity to ask me questions during my office hours. I *strongly encourage* you to take advantage of these times I make myself available! From my experience, students who attend office hours reguarly tend to learn more from the course, overall. I will happily answer any questions you have about lectures, text materials, homework problems (both present and past), past exams, general mathematics, or anything else that might come up. I will also have a Teaching Assistant who will hold other office hours to assist you. The evening before exams, I will hold a review session to answer any lingering questions you have, as well. Finally, there is always peer tutoring available through the university's tutoring center, so feel free to seek that out.

5. Important Policies:

Attendance: Since this is a summer session, we only have *six weeks* in which to squeeze an entire semester. At times, it will feel like we are moving at a rapid pace; in fact, we will be! Accordingly, it is very important that you attend *every lecture*, since each one represents a significant portion of the course. If you must miss a lecture for an excusable reason, try to let me know at least a day in advance; we can arrange to meet and discuss what you'll miss. If you do miss a lecture for *any* reason, please don't be embarrassed to just come and talk to me; I'll be more offended if you *don't* get caught up on the material, honestly. I also expect you to show up to class *on time*, i.e. at or before 9:00 am, especially when homework is due. I will try not to embarass anyone who shows up noticeably late, but I make no promises! ☺

In-class behavior: During class, I expect your full attention. Your cell phones should be set to silent (or vibrate, if you must). You will *not* be allowed to use laptops in class, and if I see anyone using one, I will claim it as my own for the day. If you wish to be able to use one for note-taking purposes, please talk to me outside of class; only then will I refrain from borrowing your electronic equipment. On this subject, you shouldn't need a calculator for any of the arithmetic in this course, so you will *not* be allowed to use on during exams.

Academic honesty: I expect you to follow the university's standard academic honesty policies. I described my own policies about homework on the previous page, too, but I should also mention that consulting the internet for solutions constitutes cheating. Again, trust me, I will be able to tell, so don't even bother doing it. If you do happen to consult another book or a website for some guidance (looking up a definition or theorem or a result we proved in class), go ahead and note that on your homework, to be safe.

Special needs: If you require special accomodations, such as extra time or disability considerations, I will work with you closely to make sure every need is met. I only request that you approach me with the required documentation within the *first week* of class, so that we may work out any necessary details and scheduling conflicts.

Hey, what about you, Mr. Instructor?!: Finally, here's what you can expect from me. I will show up to every class on time, fully prepared to teach you everything you need to know to pass this course with flying colors and succeed in the rest of your mathematical endeavors. I will have abridged lecture notes to provide for you during each class session. Homeworks and exams will be returned to you within 48 hours (assuming you come to class to pick them up, of course). I will look at any graded assessment of yours and offer comments as to why it was graded in a particular way–and raise your grade, if it seems warranted–but I request that you bring grading issues to my attention within five days. I will answer any questions you have in class or during office hours, within reason (i.e. I won't solve your homework problems for you). I will answer any emails you send me within 24 hours, usually sooner.

Final thoughts: I truly love teaching mathematics, I love this course, and I love teaching this course. (You might be surprised that some people only like two of those three, or fewer, but it's true!) At this point, Concepts is my life; I teach it, I spend my days writing about it, and I spend my free time thinking about it. I am in a uniquely beneficial position, for both you and me. I have the opportunity to share some course materials with you and use our experiences to refine them, and you have the opportunity to work with and learn from someone who lives and breathes this course. Let's make the best of this summer session and learn

as much as we can!