# The Convergence of a Random Walk on Slides to a Presentation

Math Graduate Students

Carnegie Mellon University

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For completeness, we define a *graph* to be a pair $(V, E)$ where $V$ is a set of elements called *vertices* and $E \subseteq \binom{V}{2} = \{e \subset V : |e| = 2\}$.

We will be particularly interested in *(non-looping) directed graphs*, where the edge set $E$ is an irreflexive relation on $V$.

For the following definitions, fix a digraph with vertex set $V$ and edge relation $E$, which we call the **talk graph.**

- A **slide** is a vertex $v \in V$.
- If $v$ and $u$ are slides, and $(v, u) \in E$ then we say that $v$ is a **prerequisite** of $u$.
- A **presentation** is an walk in the underlying graph. We say that a presentation is **coherent** if it satisfies the following two properties:
  1. **Hamiltonian**
     1. **Complete**: Every slide appears in the presentation.
     2. **Non-Redundant**: No slide appears twice in the presentation.
  2. **Gradual**: If $v$ and $u$ appear in the presentation and $v$ is a prerequisite for $u$ then $v$ appears earlier.
- A talk graph is **complicated** if it had no coherent presentations.

## Theorem (Szpilrajn, 1930)

*A talk with countably many slides has at most one coherent presentation.*

- If this coherent presentation exists, it can be obtained using the following algorithm:
  1. Select the first slide which has no prerequisites among unselected slides.
  2. Add it to the presentation and repeat.
- This algorithm is not guaranteed to yield a presentation (though if it does return a presentation, it will always be coherent).
- For almost all talks, the output will contain a slide not connected in any way to the previous slide.

# The uncountable case

Szpilrajn's Theorem left the existence question open in the uncountable case.

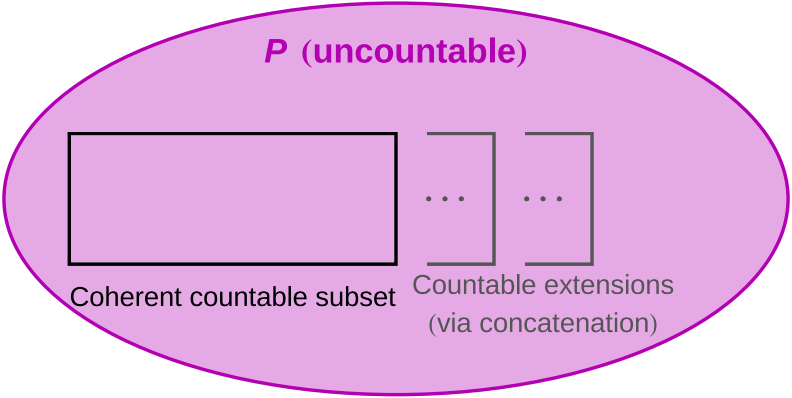
## Theorem (Natorc, 1938)

*A talk with uncountably many slides cannot have a coherent presentation.*

Roughly, the proof goes as follows: assume a coherent presentation $P$ exists.

1. Select a countable subset of slides, and assume it too has a coherent presentation. This must be a subpresentation of $P$.
2. There remains uncountably many slides to present, so one must iterate this process (use the concatenation Lemma).
3. There are only countably many *coherent* presentations. After a while, one runs out of things to say.

The proof may be visualized as follows:




A countable union of countable sets is countable, so we cannot exhaust $P$.

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**Question:** Is our theorem true without the Axiom of Choice?

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But this proof is nonconstructive. Question: Can we produce $M$ in polynomial time?

Answer:YES!

We construct a Linear Program to specify the model, $M$. The size of this LP will be polynominal in the size of $M$.

**Variables:** For each pair of elements in $M$, $A$ and $B$, we will have a variable, $x_{AB}$ which is 1, if $A \in B$, and 0 otherwise.

**Constraints:** For each axiom of ZF¬C and for our theorem, we will have a number of constraints that is polynomial in the size of $M$. (Eg. to specify that if $A \in B$ then $B \notin A$, we include the constriant $x_{AB} + x_{BA} \leq 1$)

It is obvious that these constraints form a unimodular matrix, and therefore the optimal solution has $x_{AB} \in \{0, 1\}$ for each variable $x_{AB}$.

Suppose there are $d$ possible pairs of elements $A$ and $B$ in $M$, then since $x_{AB} \in \{0,1\}$, then the optimal solution

$$X = \Pi_{A,B \in M} \{x_{AB}\}$$

is in the lattice$\{0,1\}^d$. $- \blacksquare$

## Example

In 3 dimensions, one can see that the optimal solutions are extremal points of the solution set below:

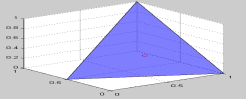


Figure: Solution set in 3 dimensions, except that 0.5 on the left should be a 0.

When it was discovered, this result lead to its author winning a Fields medal. It also lead to new research questions today such as: what happens when you let $d \to \infty$?

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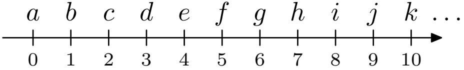

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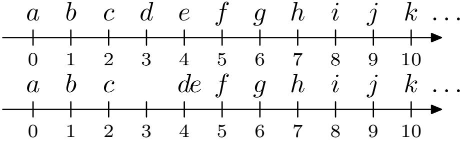

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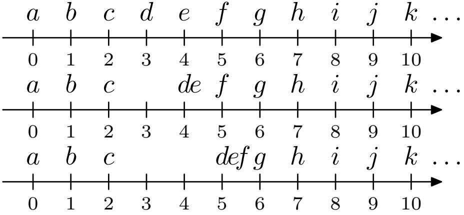

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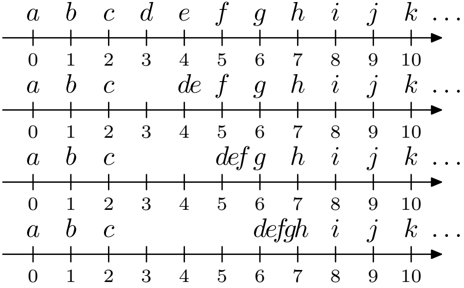

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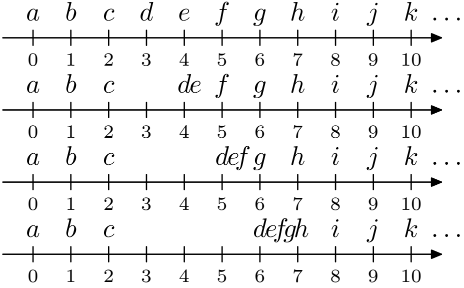


**Theorem.**

As $d \to \infty$, this process asymptotically approaches

$$O(abc) + defghijklmnopqrstuvwxyz \ldots = O(abc) + [illegible],$$

where [illegible] is the *Euler–Smasheroni constant*, [illegible] $\approx$ 0.5[illegible]...

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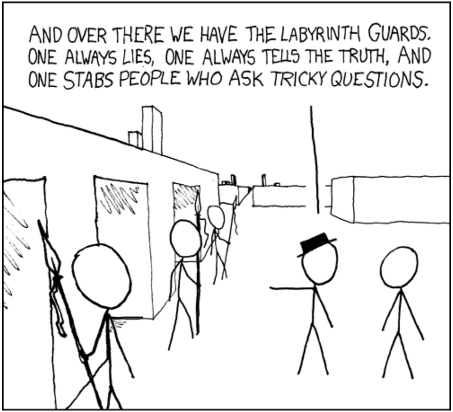

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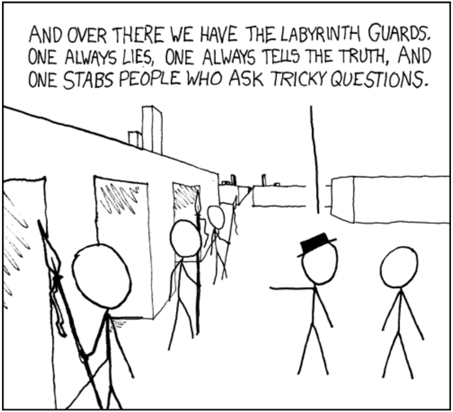


Thanks
Alot