# Memory Reduction in Iterated Function Systems

## Closing off (kind of) an avenue of measuring fractal complexity

Brendan W. Sullivan

Carnegie Mellon University
Graduate Student Seminar

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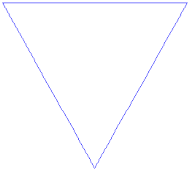
# Definitions

A fractal is . . .

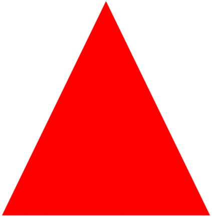
- . . . a set exhibiting self-similarity.
- . . . a set that “looks irregular; but more importantly, after it is magnified it still looks irregular.” [1]
- . . . “by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeps the topological dimension.” [2]

# Canonical examples

Figure: Simple, iteratively generated fractals



(a) Van Koch curve



(b) Sierpinski triangle (deterministic)

(c) Sierpinski triangle (probabilistic)

# Dimensional analysis

Let $X$ be a metric space. If $S \subseteq X$ and $d \in [0, \infty)$, the **$d$-dimensional Hausdorff measure of $S$** is

$$C_H^d(S) = \inf \left\{ \sum_i r_i^d \mid \exists \text{ cover of } S \text{ by balls with radii } r_i > 0 \right\}$$

and the **Hausdorff dimension** of $S$ is

$$\dim_H(S) = \inf\{d \geq 0 \mid C_H^d(S) = 0$$

The **Lebesgue covering dimension** of a topological space $X$ is the minimum $n$ such that every finite open cover $\mathcal{A}$ of $X$ admits a finite open cover which refines $\mathcal{A}$ in which no point is included in more than $n+1$ elements.

# Dimensional analysis

**Example**

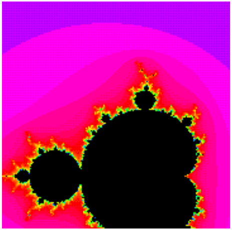
Cantor set: Hausdorff dim. $\frac{\ln 2}{\ln 3}$ ; topological dim. 0

Sierpinski $\triangle$: Hausdorff dim. $\frac{\ln 3}{\ln 2}$ ; topological dim. 1

Van Koch curve: Hausdorff dim. $\frac{\ln 3}{\ln 4}$ ; topological dim. 1

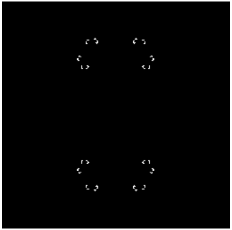
# Canonical examples

Figure: More complex, iteratively generated fractals



(a) Mandelbrot set

(b) Mandelbrot set growth

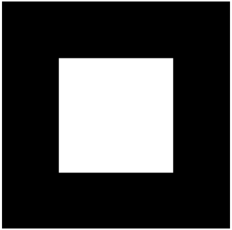


(c) Julia set

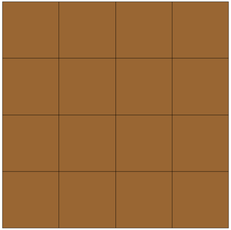
# Canonical examples

Figure: Fractals generated by iterated function systems

(a) Shrinking box border

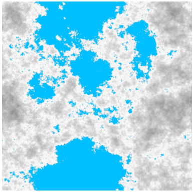


(b) Pointy leaf boxes

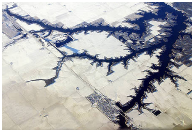


(c) Sierpinskitty trifelinegle

# Nature

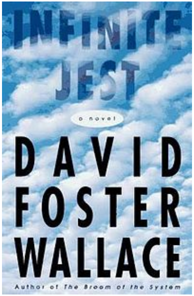

(a) Clouds [3]


(b) Plants

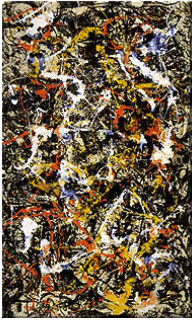

(c) Coastlines


(d) Snowflakes [4]

# Art




(a) Literature



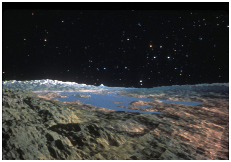
(b) Painting [5]

(c) Music [6]

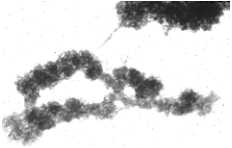
“Wind and Metal” [7]

# Science and Computing

(a) Anatomy [9]



(b) Graphics [17]



(c) DNA [18]

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**Theorem (Banach Fixed Point)**

*If $(X, \rho)$ is a complete metric space and $T : X \to X$ a contraction map, then $T$ has exactly one fixed point; i.e.*

$$\exists!\, \hat{x} \in X.\ T(\hat{x}) = \hat{x}$$

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**Question**: Does $H$ have any "fixed points"?

# Formal definition

**Partial Answer**:

**Theorem (Hutchinson, 1981)**

*For $X = \mathbb{R}^d$ with the standard metric, every IFS admits a unique compact set $A \subset \mathbb{R}^d$ satisfying $H(A) = A$.*

$A$ is the **attractor** of the IFS and is a fractal.

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**Proof.**

Show that $H$ is a contraction map on $K(X)$, the set of compact subsets of $X$. Apply Banach Fixed Point. □

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**Proof.**

Corollary to Banach Fixed Point: this limit converges to $A$ for any choice of $S_0$. □

# Examples

Standard contraction maps in $\mathbb{R}^d$ are scalings (with factor $r < 1$), rotations, reflections, translations.

# Examples

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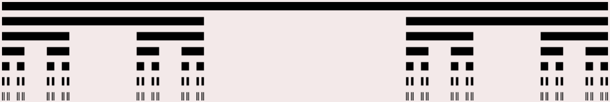
Cantor Set, $\mathcal{C}$:



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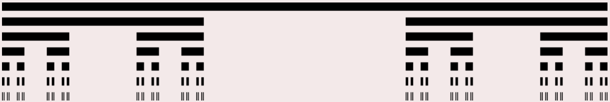

$$T_1(x) = \frac{x}{3} \qquad T_2(x) = \frac{x}{3} + \frac{2}{3}$$

$$\mathcal{C} = T_1(\mathcal{C}) \cup T_2(\mathcal{C})$$

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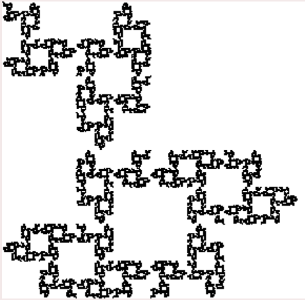
Cantor-style dust:



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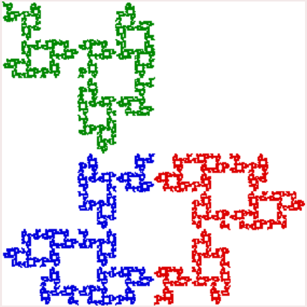

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$X = [0,1]^2$ with standard metric and $\mathcal{T} = \{T_1, T_2, T_3, T_4\}$, where

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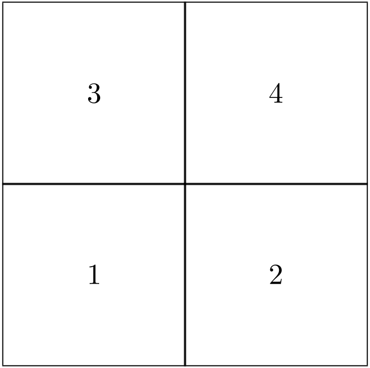

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# Standard context

Being **in state** $i$ means applying $T_i$
**Addresses** indicate the *reverse* order of the transformations required to land in that box.

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- Map composition: $(T_2 \circ T_1 \circ T_4)(X)$
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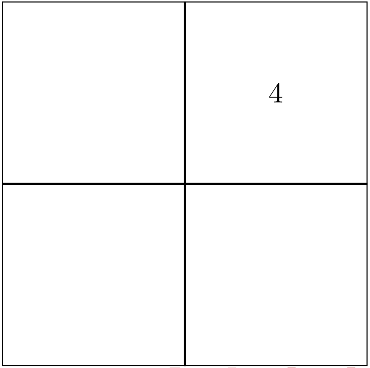

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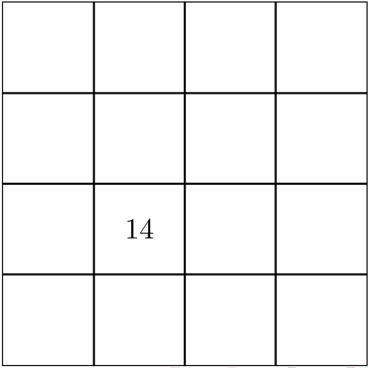

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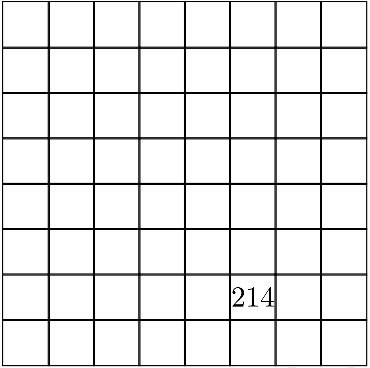

# Forbidden pairs: 1-IFS

**Main idea**: Restrict the constructive approach by disallowing certain pairs of transformations from occurring consecutively.

The system has "1 level of memory" because it looks at the currently-applied transformation in the construction to determine which transformations can be applied next.

Where you are (but not how you got there) affects where you are allowed to go.

# Forbidden pairs: 1-IFS

**Questions**:

- What types of attractors does this yield?
- How does forbidding multiple pairs affect the attractors?
- Can we look at an attractor and determine which pairs were forbidden?
- Which attractors can be realized as a *standard* IFS (with "0 levels of memory") by redefining the set of contraction maps?
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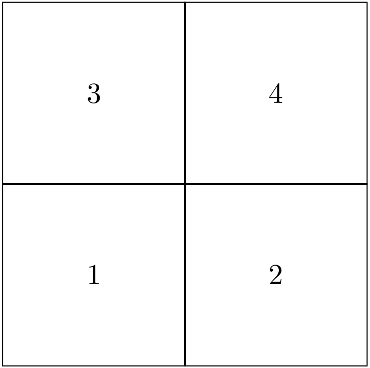


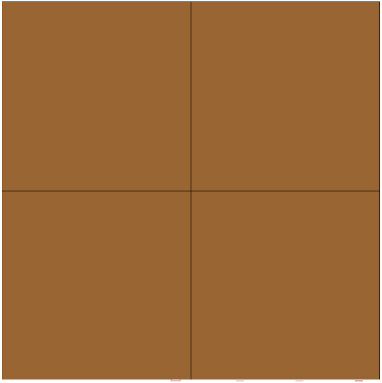

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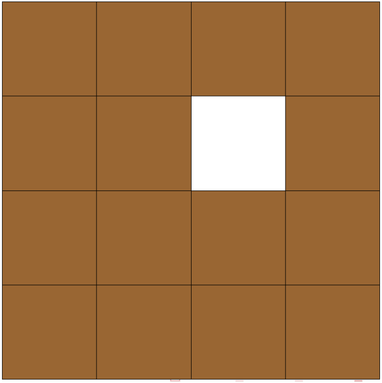

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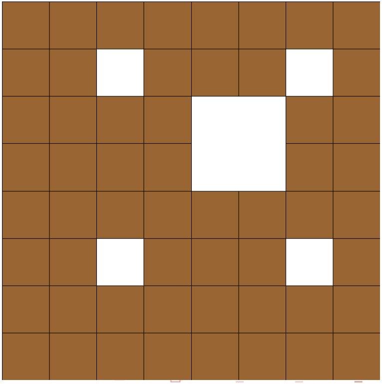

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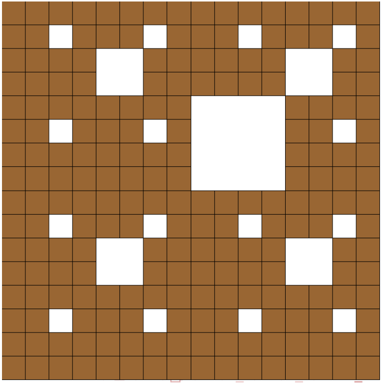

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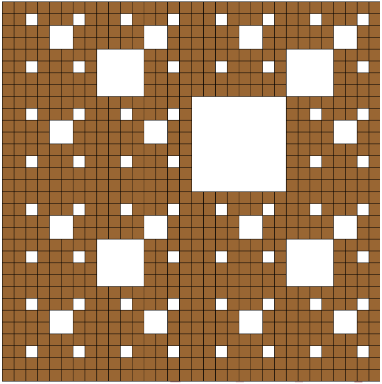

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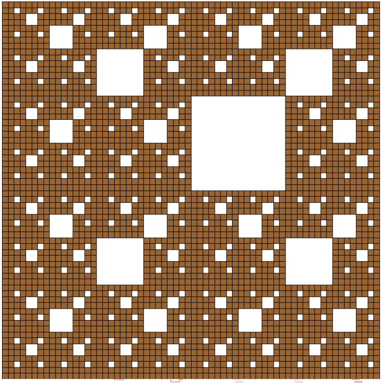
| 333 | 334 | 343 | 344 | 433 | 434 | 443 | 444 |
| 331 | 332 | | 342 | 431 | 432 | | 442 |
| 313 | 314 | 323 | 324 | | | 423 | 424 |
| 311 | 312 | 321 | 322 | | | 421 | 422 |
| 133 | 134 | 143 | 144 | 233 | 234 | 243 | 244 |
| 131 | 132 | | 142 | 231 | 232 | | 242 |
| 113 | 114 | 123 | 124 | 213 | 214 | 223 | 224 |
| 111 | 112 | 121 | 122 | 211 | 212 | 221 | 222 |



# Forbidden pairs: 1-IFS

**Example**

Any address with 41 as a substring is empty

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 333 | 334 | 343 | 344 | 433 | 434 | 443 | 444 |
| 331 | 332 | | 342 | 431 | 432 | | 442 |
| 313 | 314 | 323 | 324 | | | 423 | 424 |
| 311 | 312 | 321 | 322 | | | 421 | 422 |
| 133 | 134 | 143 | 144 | 233 | 234 | 243 | 244 |
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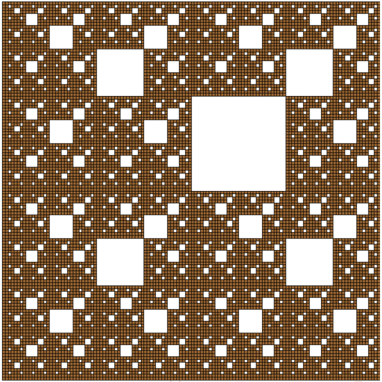

# Forbidden pairs: 1-IFS

**Example**

Any address with 41 as a substring is empty

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 333 | 334 | 343 | 344 | 433 | 434 | 443 | 444 |
| 331 | 332 | | 342 | 431 | 432 | | 442 |
| 313 | 314 | 323 | 324 | | | 423 | 424 |
| 311 | 312 | 321 | 322 | | | 421 | 422 |
| 133 | 134 | 143 | 144 | 233 | 234 | 243 | 244 |
| 131 | 132 | | 142 | 231 | 232 | | 242 |
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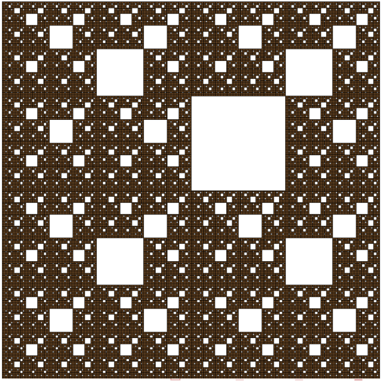

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
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| 331 | 332 | | 342 | 431 | 432 | | 442 |
| 313 | 314 | 323 | 324 | | | 423 | 424 |
| 311 | 312 | 321 | 322 | | | 421 | 422 |
| 133 | 134 | 143 | 144 | 233 | 234 | 243 | 244 |
| 131 | 132 | | 142 | 231 | 232 | | 242 |
| 113 | 114 | 123 | 124 | 213 | 214 | 223 | 224 |
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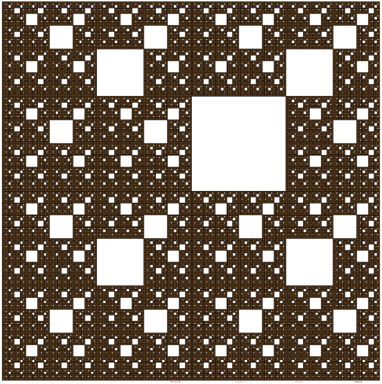


# Forbidden pairs: 1-IFS

**Example**

Forbid $1 \to 4$, $4 \to 1$, $2 \to 3$, $3 \to 2$

# Forbidden pairs: 1-IFS

**Example**

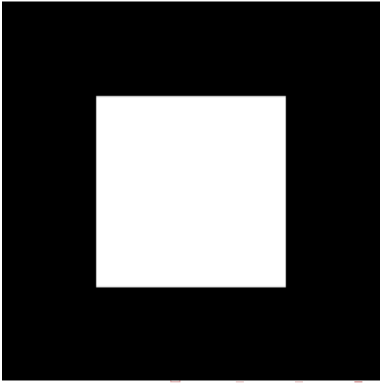
Addresses with 14, 41, 23, 32 are empty

# Forbidden pairs: 1-IFS

**Example**

Addresses with 14, 41, 23, 32 are empty

| | | | |
|---|---|---|---|
| 33 | 34 | 43 | 44 |
| 31 | | | 42 |
| 13 | | | 24 |
| 11 | 12 | 21 | 22 |



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|---|---|---|---|---|---|---|---|
| 333 | 334 | 343 | 344 | 433 | 434 | 443 | 444 |
| 331 | | | 342 | 431 | | | 442 |
| 313 | | | | | | | 424 |
| 311 | 312 | | | | | 421 | 422 |
| 133 | 134 | | | | | 243 | 244 |
| 131 | | | | | | | 242 |
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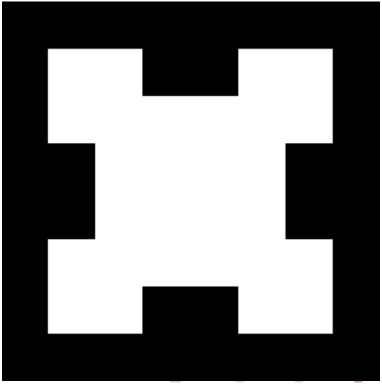

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|---|---|---|---|---|---|---|---|
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| 311 | 312 | | | | | 421 | 422 |
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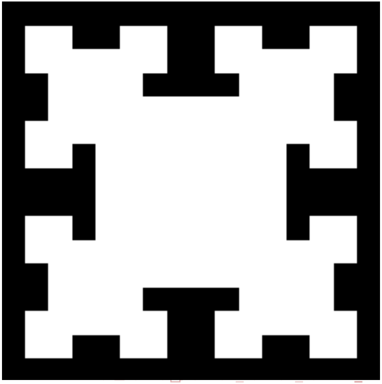

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|---|---|---|---|---|---|---|---|
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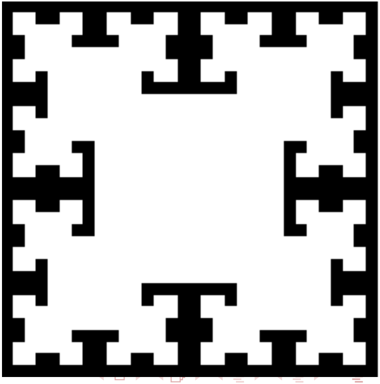

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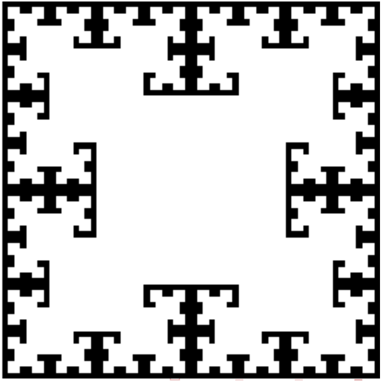

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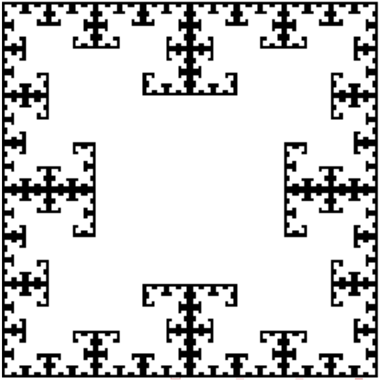

# Forbidden pairs: 1-IFS

**Example**

Addresses with 22, 23, 33 are empty

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 343 | 344 | | 434 | 443 | 444 |
| | | 341 | 342 | 431 | 432 | 441 | 442 |
| 313 | 314 | | 324 | 413 | 414 | | 424 |
| 311 | 312 | 321 | | 411 | 412 | 421 | |
| | 134 | 143 | 144 | | | 243 | 244 |
| 131 | 132 | 141 | 142 | | | 241 | 242 |
| 113 | 114 | | 124 | 213 | 214 | | |
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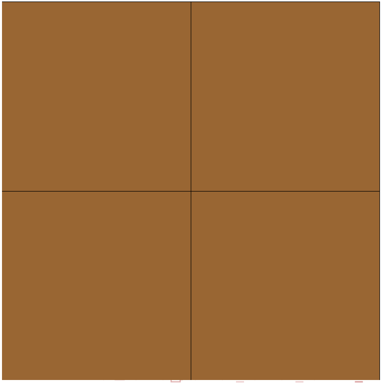

# Forbidden pairs: 1-IFS

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Addresses with 22, 23, 33 are empty

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 343 | 344 | | 434 | 443 | 444 |
| | | 341 | 342 | 431 | 432 | 441 | 442 |
| 313 | 314 | | 324 | 413 | 414 | | 424 |
| 311 | 312 | 321 | | 411 | 412 | 421 | |
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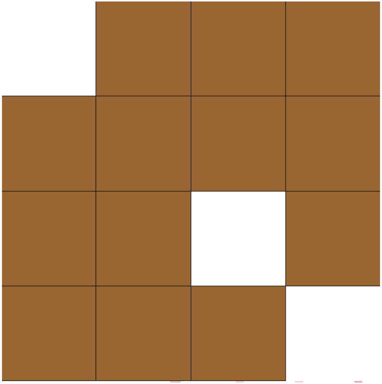

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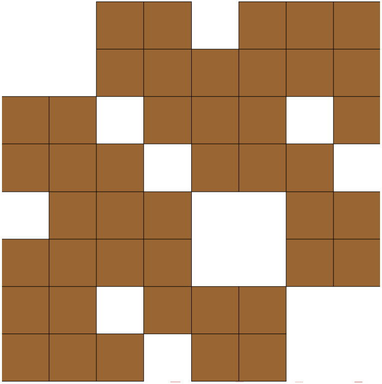

# Forbidden pairs: 1-IFS

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 343 | 344 | | 434 | 443 | 444 |
| | | 341 | 342 | 431 | 432 | 441 | 442 |
| 313 | 314 | | 324 | 413 | 414 | | 424 |
| 311 | 312 | 321 | | 411 | 412 | 421 | |
| | 134 | 143 | 144 | | | 243 | 244 |
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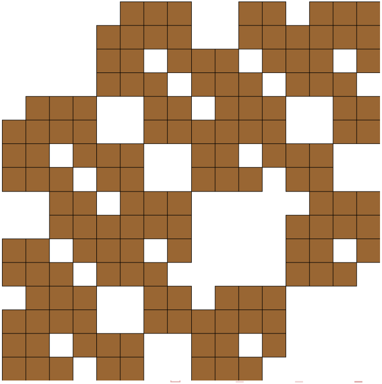

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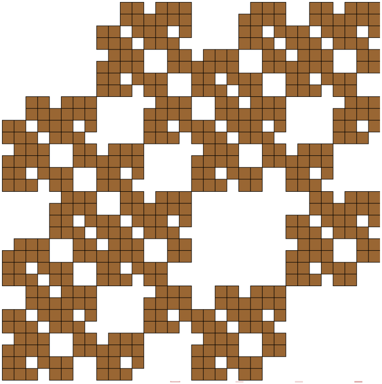

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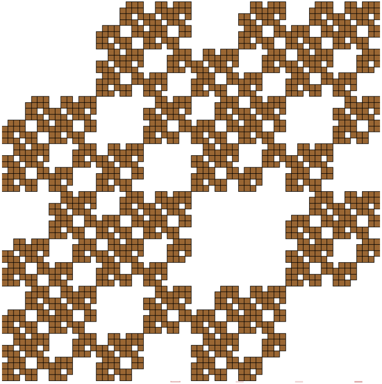

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|---|---|---|---|---|---|---|---|
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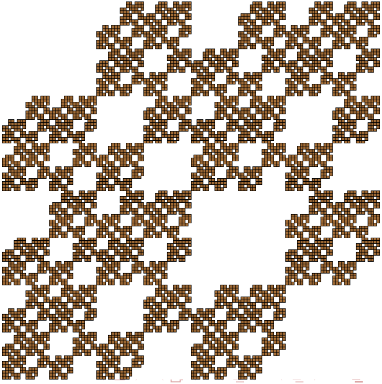

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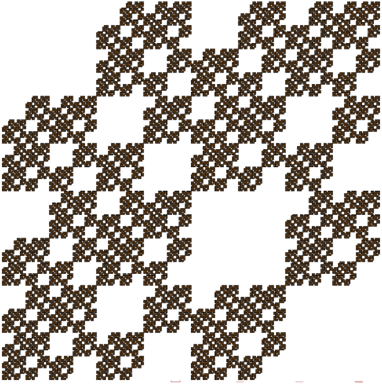

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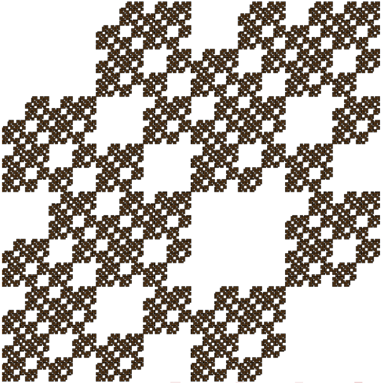

...

# Representing allowed transitions

Vertex set is $\mathcal{T}$.
Directed edge from $T_i$ to $T_j$ if $i \to j$ is allowed.

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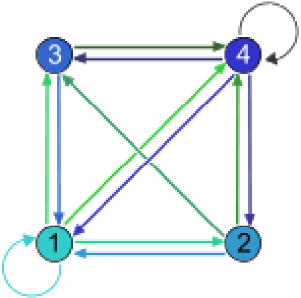
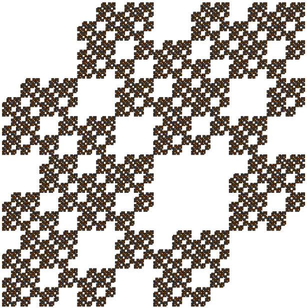
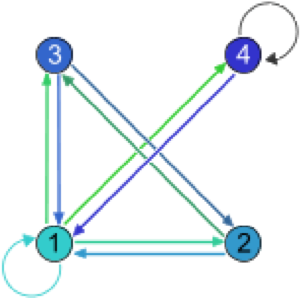
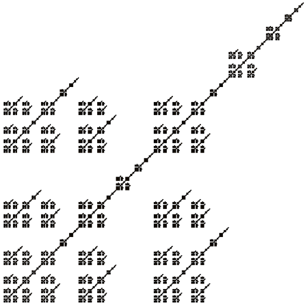


# Representing allowed transitions

Vertex set is $\mathcal{T}$.
Directed edge from $T_i$ to $T_j$ if $i \to j$ is allowed.



# Transition matrices

Represent directed edges by a 0-1 matrix.
Rows/columns indexed by *states*.
$M_{ij} = 1 \iff j \to i$ is allowed.

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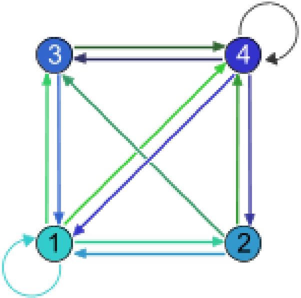


$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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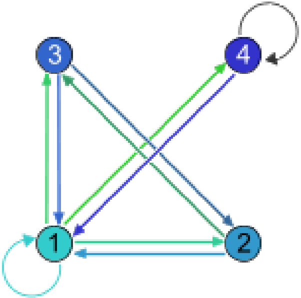


$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

# Dimensional analysis

Can use transition matrix $M$ to compute Hausdorff dimension of the attractor $A$.

Let $r_j$ be the contraction factor of $T_j$.

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Can use transition matrix $M$ to compute Hausdorff dimension of the attractor $A$.
Let $r_j$ be the contraction factor of $T_j$.

**Theorem**

*The Hausdorff dimension of $A$ is the unique $d$ for which the spectral radius of*

$$M(d) = \left[m_{ij} r_j^d\right]_{ij}$$

*is exactly 1.*

Recall: the **spectral radius** of a matrix $M$ is

$$\rho(M) = \max\{|\lambda_i| \mid \lambda_i \text{ is an eigenvalue of } M\}$$

# Terminology

**Definition**

*The attractor of a 1-IFS is …*

- **IFS-able** *if it can be realized by a 0-IFS.*
- **$\infty$-IFS-able** *if it can be realized by a 0-IFS with countably-many transformations.*
- **non-IFS-able** *if it cannot be realized by any 0-IFS, regardless of how many transformations are used.*

# Terminology

**Definition**

*A transformation $T_i$ is called a* **full state** *if it can immediately follow any other transformation;*

*i.e.* $1 \to i$, $2 \to i$, $3 \to i$, $4 \to i$ *are all allowed.*

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*A transformation $T_i$ is called a* **full state** *if it can immediately follow any other transformation;*

*i.e.* $1 \to i$, $2 \to i$, $3 \to i$, $4 \to i$ *are all allowed.*

*Also known as a* **Rome**, *because all roads in the transition graph lead to it. (Not to be confused with a* **roam**.*)*

*Corresponds to a row of 1s in transition matrix.*

# Main result: 1-IFS to 0-IFS

1. There exists a Rome.
2. Every transformation has a path to it starting at a Rome.
3. There are infinite sequences of non-Romes.

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*The attractor of a 1-IFS is ...*

- *IFS-able* $\iff$ *(1) and (2) hold.*
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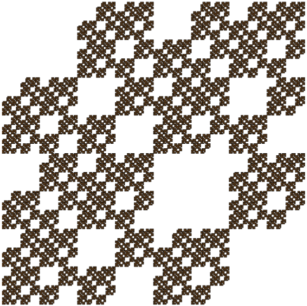
**Proof.**

*Manipulating strings and addresses ... [13]* □

# Classifying examples

**Main idea**: look for scaled copies of the attractor within itself.



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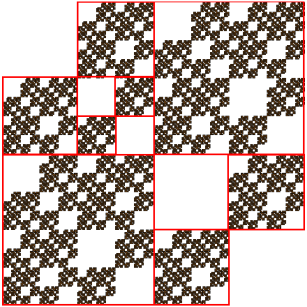
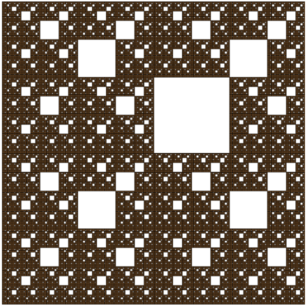


**IFS-able** with 8 transformations

# Classifying examples

**Main idea**: look for scaled copies of the attractor within itself.



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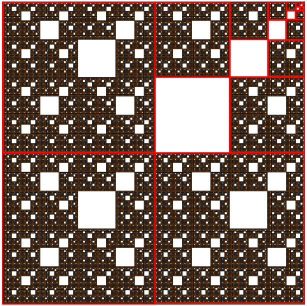

**∞-IFS-able**

# Classifying examples

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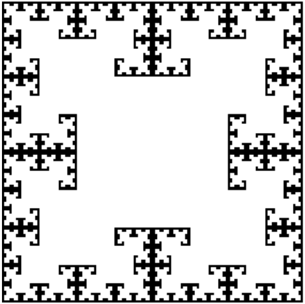

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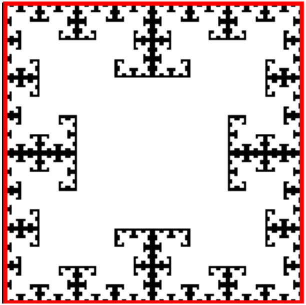


**non-IFS-able**

# Forbidden pairs and triples: 2-IFS

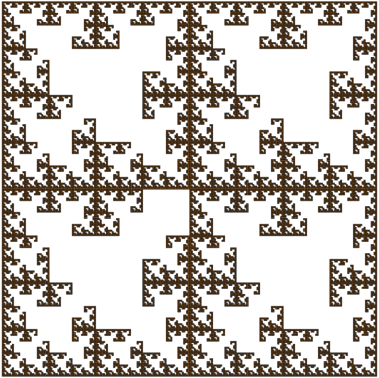
Define IFS by set $\mathcal{F}$ of forbidden strings.

2-IFS means $\mathcal{F}$ contains triples and pairs.

In general, $n$-IFS means $\mathcal{F}$ contains strings with length at most $n+1$, and contains at least one with exactly that length.

# Forbidden pairs and triples: 2-IFS

**Working example**: $\mathcal{F} = \{14, 23, 32, 441\}$



# Subshifts of finite type

Let $A$ be a finite alphabet. Let $X$ be the set of bi-infinite strings from $A$, called the **full shift**:

$$X = A^{\mathbb{Z}} = \{(\dots x_{-1}.x_0 x_1 \dots) \mid x_i \in A\}$$

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Given set of words $\mathcal{F}$ from $A$, the **shift space determined by** $\mathcal{F}$ is the set of strings from $X$ that contain no element of $\mathcal{F}$ as a substring, written as $X_{\mathcal{F}}$.

If $\mathcal{F}$ is finite, $X_{\mathcal{F}}$ is a **subshift of finite type**.

If the longest string in $\mathcal{F}$ has length $N+1$, say $X_{\mathcal{F}}$ is an $N$**-step shift**.

# Graph vertex shifts

Let $G$ be a directed graph. The **vertex shift** of $G$ has alphabet $A = V$ (vertices of $G$) and is

$$\hat{X}_G = \{ \dots v_1 . v_0 v_1 \cdots \mid \forall i.\ (v_i, v_{i+1}) \in E \}$$

the set of all infinite paths in $G$.

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**Lemma**

*Graph vertex shifts are 1-step shifts of finite type.*

**Goal**: Exploit graph shifts to reduce $n$-IFS to 1-IFS.

# Higher block shifts

Given $A, \mathcal{F}$, and $X_{\mathcal{F}}$ (an $N$-step shift).

Let $B_N(X_{\mathcal{F}})$ be the set of allowed strings of length $N$ in $X_{\mathcal{F}}$.

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Let $B_N(X_{\mathcal{F}})$ be the set of allowed strings of length $N$ in $X_{\mathcal{F}}$.

Let $\beta_N : X_{\mathcal{F}} \to (B_N(X_{\mathcal{F}}))^{\mathbb{Z}}$ be defined by

$$(\beta_N(\underline{x}))_i = x_i x_{i+1} \ldots x_{i+N-1}$$

where $\underline{x} = \ldots x_{-1}.x_0 x_1 \ldots$.

# Higher block shifts

**Example**

Let $A = \{1, 2, 3, 4\}$, $\mathcal{F} = \{14, 23, 32, 441\}$ (i.e. $N = 2$).

Consider $\underline{x} = \ldots 12443 \ldots$

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**Example**

Let $A = \{1, 2, 3, 4\}$, $\mathcal{F} = \{14, 23, 32, 441\}$ (i.e. $N = 2$).

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$$\beta_2(\underline{x}) = \ldots 12$$

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Consider $\underline{x} = \ldots 12443 \ldots$

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# Higher block shifts

**Definition**

*The* $N$**-th higher block shift** *is the image*

$$X_{\mathcal{F}}^{[N]} = \beta_N(X_{\mathcal{F}})$$

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$$X_{\mathcal{F}}^{[N]} = \beta_N(X_{\mathcal{F}})$$

**Theorem**

*If* $X_{\mathcal{F}}$ *is an* $N$*-step shift, then there exists a directed graph* $G$ *such that*

$$X_{\mathcal{F}}^{[N]} = \hat{X}_G$$

*i.e.* $N$*-th higher block shift can be realized as a 1-step shift!*

# Higher block shifts as graph shifts

Proof is in [14].
Vertices of $G$ are $B_N(X_{\mathcal{F}})$, the allowed $N$-length strings.
Edge from $a_1 \dots a_N$ to $b_1 \dots b_N$ iff

1. $a_2 \dots a_N = b_1 \dots b_{N-1}$
2. $a_1 a_2 \dots a_N b_N$ is an allowed string in $X_{\mathcal{F}}$

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Condition (1) ensures correct overlap.
Condition (2) ensures overlap is allowed.

Directed graph encodes which $N$-length strings of $X_{\mathcal{F}}$ can follow one another; i.e. it encodes which sequences of transformations are allowed by considering longer strings as the most basic elements.

# Reducing 2-IFS to 1-IFS: $\mathcal{F} = \{14, 23, 32, 441\}$

$X_{\mathcal{F}}$ is all strings from $I = \{1, 2, 3, 4\}$ without a substring in $\mathcal{F}$.

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$X_{\mathcal{F}}$ is all strings from $I = \{1, 2, 3, 4\}$ without a substring in $\mathcal{F}$.

$X_{\mathcal{F}}^{[2]}$ has alphabet $J$ consisting of 13 allowed pairs:

$$J = \{11, 12, 13, 21, 22, 24, 31, 33, 34, 41, 42, 43, 44\}$$

Construct directed graph with $J$ as vertex set.

Encode edges via transition matrix.

Ensure conditions (1) and (2) from Theorem in [14] are satisfied, i.e. *correct* and *allowed* overlaps.

# Reducing 2-IFS to 1-IFS: $\mathcal{F} = \{14, 23, 32, 441\}$

$J = \{11, 12, 13, 21, 22, 24, 31, 33, 34, 41, 42, 43, 44\}$
Column is source of edge, row is target.

$M_{ij,km} = 1 \iff i = m$ and $kij$ is allowed.

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$M_{ij,km} = 1 \iff i = m$ and $kij$ is allowed.

Overlap conditions yield many 0 entries.
(Only 41/169 nonzero entries in this example.)
Helpful in computational applications, such as computing Hausdorff dimension, powers of transition matrix, etc.

## Demonstration

$M_{ij,km} = 1 \iff m = i$ and $kij$ is allowed. Recall $\mathcal{F} = \{14, 23, 32, 441\}$

$M =$

| | 11 | 12 | 13 | 21 | 22 | 24 | 31 | 33 | 34 | 41 | 42 | 43 | 44 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 21 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 31 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 34 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 41 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |

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| 21 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 31 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 34 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 41 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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| 22 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
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| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |

# Reducing 2-IFS to 1-IFS: $\mathcal{F} = \{14, 23, 32, 441\}$

Recall $J = \{11, 12, 13, 21, 22, 24, 31, 33, 34, 41, 42, 43, 44\}$.

Define $\mathcal{F}'$ to be the forbidden pairs from alphabet $J$:

$$\mathcal{F}' = \{ij, km \in J \mid M_{ij,km} = 0\}$$

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Define $\mathcal{F}'$ to be the forbidden pairs from alphabet $J$:

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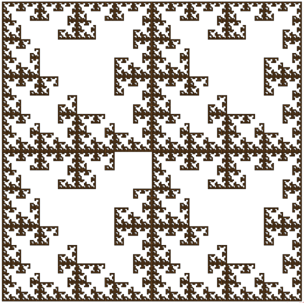
The same attractor is realized from this 1-IFS, $J(\mathcal{F}')$!



# Reducing $n$-IFS to 1-IFS

Generalizing this procedure, any $n$-IFS (forbidden strings of length $\leq n+1$) can be reduced to a 1-IFS (forbidden pairs only):

1. List all allowed strings of length $n$
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3. Apply constructive procedure to this new IFS

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**Theorem**

*The $n$-IFS $I(\mathcal{F})$ and the 1-IFS $J(\mathcal{F}')$ have the same attractor.*

**Proof.**

Compare addresses of attractors, rewrite as $I$-strings. [11] □

# Reducing $n$-IFS to 1-IFS

**Observations**:

- Procedure doesn't reduce 1-IFS to 0-IFS.
  Overlap conditions are vacuous.
- Previous results in [13] still helpful.
- Can now characterize all $n$-IFS as IFS-able / $\infty$-IFS-able / non-IFS-able

# Reducing $n$-IFS to 1-IFS

**Observations**:

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- Previous results in [13] still helpful.
- Can now characterize all $n$-IFS as IFS-able / $\infty$-IFS-able / non-IFS-able
- If $\mathcal{F}$ contains strings of length $n+1$, alphabet $J$ may have up to $4^n$ elements!

**Question**: What is the most *efficient* memory reduction procedure, yielding the least number of transformations?

# Efficient memory reduction: $\mathcal{F} = \{14, 23, 32, 441\}$

Notice 441 is a *primary string*, i.e. it does not contain a forbidden substring.

# Efficient memory reduction: $\mathcal{F} = \{14, 23, 32, 441\}$

Subdivide $T_4$ into four transformations. Define new set $\mathcal{S}$ by

$$S_1 = T_1$$
$$S_2 = T_2$$
$$S_3 = T_3$$
$$S_4 = T_4 \circ T_1$$
$$S_5 = T_4 \circ T_2$$
$$S_6 = T_4 \circ T_3$$
$$S_7 = T_4 \circ T_4$$

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\end{aligned}
$$

$$\mathcal{F}_{\mathcal{S}} = \{14, 15, 16, 17, 23, 32, 44, 45, 46, 47, 53, 62, 71, 74\}$$

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S_7 &= T_4 \circ T_4
\end{aligned}
$$

$$
M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1
\end{array}\right]
\end{array}
$$

$$\mathcal{F}_{\mathcal{S}} = \{14, 15, 16, 17, 23, 32, 44, 45, 46, 47, 53, 62, 71, 74\}$$

# Efficient memory reduction: 2-IFS to 1-IFS

**Conjecture**

*Given 2-IFS, $I(\mathcal{F})$, efficiently equivalent 1-IFS is generated by*

1. *Remove non-primary strings from $\mathcal{F}$*
2. $\forall ijk \in \mathcal{F}$, *subdivide $i$ and $j$*
3. *Reduce forbidden $ijk$; 2 cases on whether $k$ subdivided*
4. *Reduce forbidden $ij$; 4 cases on whether $i, j$ subdivided*
5. *Reduce forbidden $i$; remove compositions*

*Cases determine how many transformations needed in total.*

To be proved and investigated for $n$-IFS in [15].

# Results

- Can reduce any $n$-IFS to 1-IFS, perhaps efficiently.
- Can classify any $n$-IFS as IFS-able or not.
- Can apply method of [16] to calculate Hausdorff dimension.

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- Can classify any $n$-IFS as IFS-able or not.
- Can apply method of [16] to calculate Hausdorff dimension.
- Know that memory length is *not* a measure of fractal complexity.

# Lingering questions

- Is the efficient procedure correct?
- Is the efficient procedure actually helpful in applications?

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- Is the efficient procedure correct?
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- What exactly is the trade-off between memory length and # of transformations? Are certain formulations best for different applications?
- How many memory reductions are there with a fixed # of transformations?
- What is the relationship between $m$-IFS and $n$-IFS? Are there embeddings? Is calculting Hausdorff dimension easier in certain settings? (Partially investigated in [12])

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# THANK YOU

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