# Textbook and course materials for 21-127 Concepts of Mathematics

## Thesis defense, D.A. in Mathematical Sciences

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# Abstract

Concepts of Mathematics (21-127 at CMU) is a course designed to introduce students to the world of abstract mathematics, guiding them from more calculation-based math (that one might learn in high school) to higher mathematics, which focuses more on abstract thinking, problem solving, and writing **proofs**. This transition tends to be a shock: new notation, terminology, and expectations (particularly the requirement of **writing** mathematical thoughts formally) give students much trouble. Many standard texts for this course are found to be too dense, dry and formal by the students. Ultimately, this means these texts are unhelpful for student learning, despite their intentions. For this project, I have written a new textbook designed specifically for this course and the way it has been taught. My emphases—writing more informal prose, properly motivating all new material, and including a wider variety of exercises and questions for the reader—all reflect the goal of increasing the reader's interest and potential for learning. I have also included accompanying class notes, meant to supplement the teaching of this course, as well as assignments and exams, and their corresponding solutions and grading rubrics.

# Acknowledgements

This is joint work with Profs. John Mackey and Jack Schaeffer.

Thanks to John, Jack, Bill, and Hilary for guidance.

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# Course description

## 21-127 Concepts of Mathematics

All Semesters: 10 units

This course introduces the basic concepts, ideas and tools involved in doing mathematics. As such, its main focus is on presenting informal logic, and the methods of mathematical proof. These subjects are closely related to the application of mathematics in many areas, particularly computer science. Topics discussed include a basic introduction to elementary number theory, induction, the algebra of sets, relations, equivalence relations, congruences, partitions, and functions, including injections, surjections, and bijections. A prerequisite for 15-211. (3 hrs. lec., 2 hrs. rec.)

http://coursecatalog.web.cmu.edu/melloncollegeofscience/departmentofmathematicalsciences/courses/

# Transition from computation to proof

- High school math courses emphasize formulaic, rote application of methods: *"Follow this example"*.
- Even college calculus is focused on getting the right answer by standard techniques: *"Plug and chug"*.
- This is **not** what mathematics is truly about!
- We want students to be able to think abstractly, understand and generalize methods, explore various problem-solving techniques, and ultimately write **proofs**.
- This is a difficult transition for the students, and places a burden on the instructor and TAs.

# Who takes this course, and why?

- All years, but mostly freshmen/sophomores.
  (Have had grads and advanced high-schoolers, too.)
- All majors, but mostly MCS and CIT.
- Pre-req for most higher math courses.
  Required for math majors.
  Co-req for ECE 18-240.
  Pre-req for CS 15-150, co-req for 15-122.
- AP/EA students who want to break the "calculus cycle".
- From our perspective: We want students to understand what math really is and what mathematicians do.
- Reputation: Challenging but rewarding course. Some students take it several times.

# Material covered

"Topics discussed include a basic introduction to elementary number theory, induction, the algebra of sets, relations, equivalence relations, congruences, partitions, and functions, including injections, surjections, and bijections."

("The easier it sounds, the harder it is.")

- This allows for some variability, by instructor.
- Main goal is to introduce **proof methods** and illustrate them by exploring different topics.
- Certain fundamental topics need to be introduced here so that students are familiar with them later, e.g. induction, sets, functions.
- Other topics can be chosen and developed based on interest or timing, e.g. cardinality, number theory, probability.

# Sequencing of material

(1) Mathematical arguments
(2) Induction
(3) Sets
(4) Formal logic
(5) Proof techniques
(6) Relations
(7) Equivalence classes
(8) Modular arithmetic
(9) Number theory
(10) Functions
(11) Cardinality
(12) Combinatorics
(13) Probability
(14) Basic graph theory

These can be reordered and/or dropped in some ways, but the topics in the left column should precede those on the right.

Timing of the semester and pace of the course affects the level of detail of included topics, as well.

# Organization and assessment

- **Three 1-hr lectures, MWF**
  Led by instructor, introduces new material and examples.
- **Two 1-hr recitations, TR**
  Led by TAs, work on problem-solving and conceptual help.

- **10 written assignments**
  Emphasis on solving problems and writing proofs.
- **3 in-class exams**
  Emphasis on content recall and writing proof details quickly ("time crunch").
- **1 final**
  Emphasis on synthesizing course content and applying methods to write proofs across areas.

# Genesis and motivation

- Served as TA for 21-127. Found myself agreeing with some students' complaints about the standard text. Started developing ways to work around these conditions to present better material and help student learning.
- Was looking for a project to devote myself to. Realized I was very interested in teaching and already felt motivated to spend lots of time on course design and materials.
- Recognized how important this course is to the math curriculum, in general. Wished I had such a course in my undergrad education.

# Teaching

- Fall 2010 — TA
- Summer I 2011 — grader
- Summer II 2011 — instructor
- Fall 2011 — TA (w/ Mackey)
- Summer 2012 — instructor
- Fall 2012 — instructor (w/ Schaeffer)

While not teaching, writing!

# Writing

- Spring 2011 — Began writing with Mackey. First three chapters took shape. Stated goals and developed rough plan for the rest. Editing and guidance from Mackey based on his experience.
- Summer 2011 — Used new materials to supplement standard text while instructing. Focused on finding good motivating examples for material, as well as writing exercises to "field test".
- Fall 2011 — Spent much time helping with logistics of course, including grading (and verifications) and student help. Wrote "recitation sheets" whose content has been repurposed. Also used some of my written problems on homeworks.
- Spring 2012 — Wrote most of the later chapters. Still deciding what to include/omit and how to sequence the material.

# Writing

- Summer 2012 — Used solely my materials while instructing. Large summer class with many AP/EA students. Realized that greater emphasis on proof techniques is needed. Wrote lecture notes and developed many examples and exercises to be used.
- Fall 2012 — Used my materials and collaborated with Prof. Schaeffer on assignments/exams. Further developed lecture and recitation notes, and exercises, which were very useful in finishing the writing of the book and providing supplemental materials.
- Spring 2013 — Took teaching experiences and written materials and finished writing book. Decided on sequencing, examples and exercises. Standardized formatting and chapter structures, created many diagrams.

# Course/book tension

Much of this project developed around my simultaneous teaching of the course, in various settings.

**Good:**

- Ability to “field test” examples, explanations and exercises.
- Instant feedback from students on materials, and given-and-take from me (via AnnotateMyPDF).

**Bad:**

- Had hopes for extra material that was not approached due to time/space constraints.
- Might limit appeal of the book to outside sources, or even other instructors.

# Textbook/informality tension

As described later, one major goal is to write less formally than other texts to engage the reader.

**Good:**

- This idea developed from direct experiences with students, and it seemed like this is what they wanted (or would unknowingly benefit from).
- Makes the book stand out amongst others; non-standard for a math book designed for budding math-doers.

**Bad:**

- Might "turn off" especially rigorous students who were already motivated to pursue higher mathematics.
- Length and "wordiness" can be overwhelming.

# Overall structure of book

- **Part I: Learning to Think Mathematically**
  - Introduction to mathematical arguments (What is a proof?) including inductive arguments
  - Sets as fundamental objects, new definitions and notation
  - Logic and proof techniques (central focus)
  - Return to induction, formalization and further development
- **Part II: Learning Mathematical Topics**
  - Applying proof techniques to other topics while using problem-solving skills and gathering knowledge
  - Relations and properties, equivalence relations and classes, modular arithmetic, number theory
  - Functions and properties, cardinality, infinite sets
  - Combinatorics, basic counting arguments, counting in two ways, intro to advanced topics

(See other handout, TOC)

# Part I

1. What is Mathematics?
2. Mathematical Induction: “And so on . . . ”
3. Sets: Mathematical Foundations
4. Logic: The Mathematical Language
5. Rigorous Mathematical Induction: A Formal Restatement

# Part II

6. Relations and Modular Arithmetic: Structuring Sets and Proving Facts About The Integers
7. Functions and Cardinality: Inputs, Outputs, and The Sizes of Sets
8. Combinatorics: Counting Stuff

# 1. What is Mathematics?

1.1 Present four “proofs” of the Pythagorean theorem.
Discuss prime numbers and the infinitude of primes.
Argue there exist irrational $a, b$ such that $a^b$ is rational.

1.2 Discuss importance of notation and definitions.
Informally discuss logic and show *bad/incorrect* proofs.
Dicuss some history and preview chapter on logic.

1.3 Describe presumed knowledge: familiarity with arithmetic and high-school level algebra (showing practice examples).
Solve systems of equations, derive Quadratic Formula.
Discuss standard sets of numbers, preview chapter on sets.

# 1. What is Mathematics?

1.4 Present four interesting *puzzles*. Thoroughly discuss solutions, as well as how to approach them. Summarize each with a lesson.

- The Missing Dollar Problem
- Gauss' Sum: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
- Sum of odds: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$
- Monty Hall Problem

1.5 Exercises: Algebraic problems, simple puzzles that require some ingenuity, practice with definitions and notation.

1.6 Lookahead: Focus on the sum puzzles, inductive arguments.

# 2. Mathematical Induction

2.1 Intro: objectives, segue, motivation, goals and warnings.

2.2 Examples of "regular induction": $\sum k^2$ and lines on a plane

2.3 Domino Analogy (and others), an informal discussion

2.4 Examples of "strong induction": Fibonacci tilings and "Takeaway" (basic Nim, handout 1, p. 131)

2.5 Applications: recursive programming, Tower of Hanoi

2.6 Summary

2.7 Exercises: inductive arguments, guiding through induction proofs, discovery of formulae, several "spoofs"

2.8 Lookahead: develop the fundamentals to be more rigorous

# 3. Sets

3.1 Intro: objectives, segue, motivation, goals and warnings.

3.2 Main idea: examples from everyday life and math

3.3 Definition and examples: proper notation, set-builder usage, empty set, Russell's Paradox (and a brief note on axioms)

3.4 Subsets: definition and standard examples, finding the power set, equality by containment, the "bag analogy"

3.5 Set operations: $\cap, \cup, -, \overline{A}$ (handout 4, p. 174)

3.6 Indexed sets: notation, usage, examples, operations

3.7 Cartesian products: ordered pairs, examples

3.8 [Optional] Defining $\mathbb{N}$ via sets: inductive sets, state PMI

# 3. Sets

3.9 Proofs involving sets:
Introduces idea of *appealing to formal definitions*.
How to prove $\subseteq$
Double-containment proofs to show $=$
Many examples, including indexed operations
Disproving claims (introduces logical negation gently)

3.10 Summary

3.11 Exercises: practice with notation and reading statements, asks reader to provide examples and non-examples, several "spoofs", proofs involving sets (*avoiding* arguments that would be made easier via logic, e.g. DeMorgan's Laws)

3.12 Lookahead: logical ideas, develop proof techniques to apply

# 4. Logic

4.1 Intro: objectives, segue, motivation, goals and warnings.

4.2 Mathematical statements: variable propositions, examples and non-examples, proper notation

4.3 Quantifiers: usage and notation, how to read *and* write statements, "fixed" variables

4.4 Negating quantifiers: method and examples, Law of the Excluded Middle, redefine indexed set operations

4.5 Connectives: $\wedge, \vee, \implies, \iff$
Many examples and non-examples of each, in-depth discussion of $\implies$ and various forms, redefine set operations

# 4. Logic

4.6 Logical equivalence: definition and usage and examples, biconditionals, necessary/sufficient, associative/distributive laws, DeMorgan's Laws, (double-)containment proofs

4.7 Logical negations: use DeMorgan's Laws, negating $P \implies Q$, method and examples

4.8 [Optional] Truth sets: relating connectives to sets

4.9 Proof strategies: major focus, outline direct/indirect/other methods for each connective, implement an example showing necessary scratch work (handout 3, p. 286)

4.10 Summary

4.11 Exercises: applying proof techniques, discovering truths

4.12 Lookahead: have learned ideas to revisit induction

# 5. Formal Mathematical Induction

5.1 Intro: objectives.

5.2 Regular induction: state and prove PMI, provide proof template and illustrate usage, emphasize common errors

5.3 Variants: different base case, backwards, evens/odds

5.4 Strong induction: state and prove PSMI, provide proof template and illustrate usage, compare to PMI

5.5 Variants: "minimal criminal", WOP, TFAE

5.6 Summary

5.7 Exercises: more difficult arguments, prove WOP, "spoofs"

5.8 Lookahead: new goal, functions

This concludes Part I (just over halfway).

# 6. Relations and Modular Arithmetic

6.1 Intro: objectives, segue, motivation, goals and warnings.

6.2 Binary relations: definition and examples, properties (reflexive, symmetric, transitive, anti-symmetric) and canonical examples, proving/disproving properties

6.3 [Optional] Order relations: posets, tosets, chains, (to include: well orders)

6.4 Equivalence relations: examples and motivation, equivalence classes and how to characterize them, partitions, theorems (some proofs as exercises)

# 6. Relations and Modular Arithmetic

6.5 Modular arithmetic: equivalence classes mod $n$, using mods to prove facts, multiplicative inverses (relatively prime), solving Diophantine equations, CRT, Bézout

6.6 Summary

6.7 Exercises: proving properties of relations, characterizing equivalence classes, solving number theory claims, proving interesting lemmas and theorems, some “spoofs”

6.8 Lookahead: a function is a relation

# 7. Functions and Cardinality

7.1 Intro: objectives, segue, motivation, goals and warnings.

7.2 Definition and examples: “well-defined”, functional equality, schematics, breaking idea that it’s a “rule” on numbers

7.3 Images and pre-images: definitions and easy/hard examples, proof strategies (reiterate sets and logic), constructing counterexamples (handout 2, p. 488)

7.4 Properties: “jections”, definitions and examples, proof strategies, how to determine properties

7.5 Compositions and inverses: notation and usage and examples, proving inverse by composing both ways to get identity, bijection iff invertible

# 7. Functions and Cardinality

7.6 Cardinality: finite vs. infinite, comparing via functions (discussion on axioms/defns), Cantor's Theorem, countably infinite sets (Hilbert Hotel, examples and theorems, infinite vs. arbitrarily large), uncountably infinite sets (examples and theorems, levels of infinity)

7.7 Summary

7.8 Exercises: wide range of difficulties, many lemmas from chapter, exploring properties, prove/find counterexample, "spoofs", sets of binary strings (handout 5, p. 556)

7.9 Lookahead: focus on finite sets

# 8. Combinatorics

8.1 Intro: objectives, segue, motivation, goals and warnings.

8.2 Basic counting principles: Rules of Sum and Product, fundamental objects and formulae (permutations, selections, binomial coefficients, arrangements), relation to functions

8.3 Counting arguments: combining ROS/ROP, case analysis, decision processes, being careful of under/overcounts, other objects ($n$-tuples, alphabets, anagrams, lattice paths)

8.4 Counting in two ways: method summary, examples, theorems and uses, how to analyze an identity and construct an argument

8.5 Selections with repetition: Pirates & Gold (stars & bars), balls in bins, indistinguishable dice, examples

# 8. Combinatorics

8.6 Pigeonhole: statement and proof (logic), examples

8.7 Inclusion/Exclusion: statement and proof, examples where intersections have fixed size, other examples (sels. w/rep.)

8.8 Summary

8.9 Exercises: wide variety of difficulties, many counting in two ways, some "spoofs" to identiy over/undercounts, Fermat's Little via binomial coeffs, comparing selections with and w/o repetition

8.10 Lookahead: go forth and prosper

# Appendix

Definitions and Theorems

Separated by topic (not necessarily chronology)

Proof strategies for connectives, functions, induction

Cardinality catalog

Acronyms and phrases

Helpful reference (suggested by students)

# Prose: readable, engaging, but meaningful

- See handout 1, p. 131-134.
- Encourage reader to explore the concepts and examples on their own. Ask insightful questions to guide them.
- Present motivation and analysis as if in the role of a knowledgeable fellow student, but backed by expert insight.
- Admittedly, writing looks "texty" but this is broken up by section headings, itemized lists, diagrams, and questions.
- Encouraging the reader: we are on the same journey.
- Written as if I were speaking in the classroom, but benefits from organization and foresight.
- Not an *informational* reference necessarily, but a *conceptual* reference. (Where else can students find fully explained and motivated theorems and examples?)

# Information: avoiding “expert blind-spots”

- See handout 2, p. 488-491.
- Mathematical writing tends to treat the reader as a fellow expert, with no mention of work “behind the scenes”.
- “We often skip or combine critical steps when we teach. Students, on the other hand, don’t yet have sufficient background and experience to make these leaps and can become confused, draw incorrect conclusions, or fail to develop important skills. They need instructors to break tasks into component steps, explain connections explicitly, and model processes in detail.” Eberly Center: http://www.cmu.edu/teaching/principles/teaching.html
- Somewhat like the “follow this example” method but emphasizes *critical thinking*, not a rote algorithm.
- See handout 3, p. 286-291 for a combo of these approaches.

# Consistency of chapter structures

Each chapter has the following outline:

- Introduction:
  - Objectives
  - Segue from previous chapter
  - Motivation
  - Goals and warnings for the reader
- Sections:
  - Content
  - Questions: Remind Yourself
  - Exercises: Try It
- Conclusion:
  - Summary
  - Chapter exercises
  - Lookahead

# Consistency of chapter structures

Example of objectives (from Chapter 8: Combinatorics):
**By the end of this chapter, you should be able to . . .**

- State the Rules of Sum and Product, and use and combine them to construct simple counting arguments.
- Categorize several standard counting objects, as well as state corresponding counting formulas and understand how to prove them.
- Understand the meaning of binomial coefficients, how to use them in counting arguments, and how to derive their numerical formula.
- Critique a proposed counting argument by properly demonstrating if it is an undercount or overcount.
- Prove combinatorial identities by constructing “counting in two ways” proofs.
- Understand various formulations of selection with repetition, and use them to solve problems.
- State the Pigeonhole Principle and use it in counting arguments.
- State the Principle of Inclusion/Exclusion and use it in counting arguments.

# Section exercises: easy concept checks

See handout 4, p. 174-175

- Questions: Remind Yourself
  Checking the ability to recall a definition or theorem, use proper notation, identify the difference between two concepts, name canonical examples/non-examples.
- Exercises: Try It
  Require some more thought/effort but are not meant to be too challenging. Reminds the reader they need to *do* math.
- Together, summarizing and reinforcing main ideas from the section. Building up understanding to move on with more content. Easing into more difficult chapter exercises.

# Chapter exercises: variety and synthesis

See handout 5, p. 556-565.

- Combines material from that chapter and all previous.
- Problems range widely in difficulty. (Typically, easiest ones come first; remainder are spread out.)
- Easy problems can amount to understanding definitions and notation.
- Always features a few “spoofs”: find the flaw (if any!) in a proposed argument. *Essential* mathematical skill.
- Often asks reader to prove lemmas/theorems we stated.
- Several prove/disprove problems.
- Some difficult problems *scaffolded* to guide the reader (e.g. 7.8.30, p. 561, structured double induction).
- Various hints and suggestions.

# Class notes

- Lecture notes:
  Condensed versions of definitions and examples, with some discussion. Some new examples and questions.
  Helpful for instructor to set pace/timing of course.
  Helpful for student to have (perhaps) more digestible notes.
- Recitation notes:
  Extra examples and exercises to work through.
  Supplements course material with problems that wouldn't squeeze into a lecture (time/content).
  (Two versions: one for TA notes, one for student handout.)
  Helpful for instructor/TAs to have suggested problems.
  Helpful for student to have written record of (often) predominantly verbal problem-solving sessions.

# Homeworks and exams

- Most homework problems appear in book.
  Helpful for instructor to have identified set of problems that are especially fruitful for students to work on.
- Most exam problems do *not* appear in book.
  Helpful for instructor to see what can feasibly be completed in 50 minutes. Exams carefully designed to minimize time crunch (Prep Questions) and address all required skills (concepts, read math, problem-solve, write proofs).
- Solutions and rubrics included.
  Helpful for student to have fully detailed examples of *good* proofs. Need role models.
  Helpful for instructor/TAs to have indication of what is important in a solution (based on common errors).

Why bother?

Aren't there many books/courses that do this?

I *knew* there could be a better book, one that would be more helpful for students and would bring more of them into the world of mathematics. (In particular, I wish I'd had such a book.)

"The D.A. thesis . . . [is expected] to demonstrate an ability to organize, understand, and present mathematical ideas in a scholarly way, usually with sufficient originality and worth to produce publishable work."

http://www.math.cmu.edu/graduate/PhDprogram.html

# Addresses teaching principles

Effective teaching involves . . .
http://www.cmu.edu/teaching/principles/teaching.html

- . . . acquiring relevant knowledge about students and using that knowledge to inform our course design and classroom teaching.
- . . . articulating explicit expectations regarding learning objectives and policies.
- . . . prioritizing the knowledge and skills we choose to focus on. ("Coverage is the enemy.")
- . . . recognizing and overcoming our expert blind spots.
- . . . adopting appropriate teaching roles to support our learning goals.

# Made from experience and trial

- Much of the content was developed first in recitations, seeing where students needed more/fewer examples, more/less illustration of theorems/proofs vs. applications, and gathering helpful heuristic explanations.
- This was developed into full-fledged lecture notes and textbook writing.
- Likewise, interactions in office hours, plus lots of rubric writing and grading, have informed decisions about exercises and expectations.
- Use of AnnotateMyPDF has brought suggestions and criticisms directly from a variety of readers.

  "This example was helpful." "I didn't understand this theorem until . . . "

# Stands out as different

- Casually flipping through, you can *tell* this is not a "standard" math text.
- Investigating further, you'll (hopefully) find yourself wanting to read more.
  (If you find it redundant/unnecessary, it wasn't designed for you in the first place, and would be better as a guide.)
- Math seems to make outsiders feel unworthy, or at least mightily tests their mettle. I place my writing on the level of a novice reader while simultaneously guiding them in.
- Books with similar prose style are aimed at the "amateurs" and "puzzlers", not necessarily looking to bring them into the world of abstract math.

# Potential for both self-study and use in teaching

- Engaging writing style could be perfect for a motivated reader, even without the structure of a course/instructor.
- No knowledge presumed beyond familiarity with numbers and algebra, so could easily be recommended to a developed high-schooler wondering where to go.
- Varied difficulty in exercises can appropriately test anyone from a casual reader to a devoted student.
- Breadth of examples and exercises gives instructor lots of choices. Course can follow book closely or use it as a reference for further development of material.
- Points out "lack of time/space" wherever relevant. Instructor could supplement these sections with notes/problems.

# Potential issues with both self-study and use in teaching

- Might be difficult to significantly alter sequencing of course material, e.g. formal logic entirely before sets.
- Might be difficult to significantly alter pacing of course material, e.g. length of chapters does not correlate to classroom time, and e.g. whether or not to assign readings.
- Worry that a reader will expect all texts to be like this.
- Worry that engaging style might cut down on external study and dedicated work (despite insistence otherwise).

# Readability vs. reference

- "Wordy" writing might negatively affect future use as a reference. Potential difficulties in locating material, wading through no-longer-needed explanations.
- Many important aspects of the course are *conceptual* in nature. Easy to look up definitions/theorems, but what about ideas and strategies?
- Will students who don't necessarily need the extra explanations then not bother to read, and be worse off?

# Informality can be confusing

- Potential issues with ESL students: colloquialisms, cultural references, sheer volume of text can be off-putting.
- Am I just "passing the buck" on the transitional shock to higher mathematics?
- Might this style dissuade students who were looking for cold, hard rigor from pursuing more math?

# "Sufficient originality and worth"

- Detailed learning objectives and motivation within every chapter/section.
- Vast array of exercises to engage both a struggling novice and a partial expert.
- Emphasis on *reading* in a book (how novel!).
- Written not only with the novice student as the intended audience but also ACTFTPOVAIFP.
- Content founded on rigorous mathematics and curricular goals, while presentation founded on education research.
- Ultimately, believe the positives outweight the negatives, and they can both be massaged by attentive instruction.

# What do we have now?

- Textbook designed specifically (and “play-tested”) for use in this course. Content sequencing, motivations, examples, and exercises designed from experience and usage.
- Supplemental materials to aid in the teaching and logistical implementation of this course.
- Book could serve as standalone text for motivated reader.
- Set of materials could serve as reference/inspiration for (future) instructors/TAs.

# Other ideas to explore

- Further chapters:
  Probability. Graphs/discrete structures. More advanced combinatorics. Intro to abstract algebra and groups. Intro to analysis. Point-set topology. ("Concepts II")
- Further content:
  Appendix with hints. Supplemental instructor's solutions. More exercises (particularly challenging ones).
- Applying this writing style to other courses/material:
  More apt for "lower level" math, particularly courses meant for non-majors. Could conceivably be useful on smaller scales in upper-level math, though.
  At least, rethink learning objectives and presentation style.
- Online access.

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THANK YOU

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Questions?