# Numerous Proofs of $\zeta(2) = \frac{\pi^2}{6}$

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April 15, 2013

**Abstract**

In this talk, we will investigate how the late, great Leonhard Euler originally proved the identity $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ way back in 1735. This will briefly lead us astray into the bewildering forest of complex analysis where we will point to some important theorems and lemmas whose proofs are, alas, too far off the beaten path. On our journey out of said forest, we will visit the temple of the Riemann zeta function and marvel at its significance in number theory and its relation to the problem at hand, and we will bow to the uber-famously-unsolved Riemann hypothesis. From there, we will travel far and wide through the kingdom of analysis, whizzing through a number N of proofs of the same original fact in this talk's title, where N is not to exceed 5 but is no less than 3. Nothing beyond a familiarity with standard calculus and the notion of imaginary numbers will be presumed.

**Note:** These were notes I typed up for myself to give this seminar talk. I only got through a portion of the material written down here in the actual presentation, so I figured I'd just share my notes and let you read through them. Many of these proofs were discovered in a survey article by Robin Chapman (linked below). I chose particular ones to work through based on the intended audience; I also added a section about justifying the $\sin(x)$ "factoring" as an infinite product (a fact upon which two of Euler's proofs depend) and one about the Riemann Zeta function and its use in number theory. (Admittedly, I still don't understand it, but I tried to share whatever information I could glean!)

http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf

The Basel Problem was first posed by the Italian mathematician Pietro Mengoli in 1644. His question was simple:

$$\text{What is the value of the infinite sum} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad ?$$

Of course, the connection to the Riemann zeta function came later. We'll use the notation for now and discuss where it came from, and its significance in number theory, later. Presumably, Mengoli was interested in infinite sums, since he had proven already not only that the harmonic series is divergent, but also

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$$

and that Wallis' product

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots = \frac{\pi}{2}.$$

is correct. Let's tackle the problem from the perspective of Euler, who first "solved" the problem in 1735; at least, that's when he first announced his result to the mathematical community. A rigorous proof followed a few years later in 1741 after Euler made some headway in complex analysis. First, let's discuss his original "proof" and then fill in some of the gaps with some rigorous analysis afterwards.

**Theorem 1.** $\zeta(2) = \frac{\pi^2}{6}$

*Proof #1, Euler (1735).* Consider the Maclaurin series for $\sin(\pi x)$

$$\sin(\pi x) = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{2n+1}x^{2n+1}}{(2n+1)!} =: p(x)$$

We know that the roots of $\sin(\pi x)$ are the integers $\mathbb{Z}$. For *finite* polynomials $q(x)$, we know that we can write the function as a product of linear factors of the form $(1 - \frac{x}{a})$, where $q(a) = 0$. Euler conjectured that the same trick would work here for $\sin(\pi x)$. Assuming, for the moment, that this is correct, we have

$$\begin{aligned}\hat{p}(x) :&= \pi x\left(1-\frac{x}{1}\right)\left(1+\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{3}\right)\left(1+\frac{x}{3}\right)\cdots \\ &= \pi x\left(1-\frac{x^2}{1}\right)\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)\cdots = \pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)\end{aligned}$$

Notice that we have included the leading $x$ factor to account for the root at 0, and the $\pi$ factor to make things work when $x = 1$. Now, let's examine the coefficient of $x^3$ in this formula. By choosing the leading $\pi x$ term, and then $-\frac{x^2}{n^2}$ from one of the factors and 1 from all of the other factors, we see that

$$\hat{p}(x)[x^3] = -\pi\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots\right) = -\pi\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Comparing this to the coefficient from the Maclaurin series, $p(x)[x^3] = -\frac{\pi^3}{6}$, we obtain the desired result!

$$-\frac{\pi^3}{6} = -\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

So why is it that we can "factor" the function $\sin(\pi x)$ by using what we know about its roots? We can appeal to the powerful Weierstrass factorization theorem which states that we can perform this root factorization process for any entire function over $\mathbb{C}$.

**Definition 2.** *A function* $f : D \to \mathbb{C}$ *is said to be* holomorphic *on a domain* $D \subseteq \mathbb{C}$ *provided* $\forall z \in D$ $\exists \delta$ *such that the derivative*

$$f'(z_0) = \lim_{y \to z_0} \frac{f(y) - f(z_0)}{y - z_0}$$

*exists* $\forall z_0 \in B(z, \delta)$*. A function* $f$ *is said to be* entire *if it is holomorphic over the domain* $D = \mathbb{C}$*.*

There are two forms of the theorem, and they are essentially converses of each other. Basically, an entire function can be decomposed into factors that represent its roots (and their respective multiplicities) and a nonzero entire function. Conversely, given a sequence of complex numbers and a corresponding sequence of integers satisfying a specific property, we can construct an entire function having exactly those roots.

**Theorem 3** (Weierstrass factorization theorem)**.** *Let* $f$ *be an entire function and let* $\{a_n\}$ *be the nonzero zeros of* $f$ *repeated according to multiplicity. Suppose* $f$ *has a zero at* $z = 0$ *of order* $m \geq 0$ *(where order 0 means* $f(0) \neq 0$*). Then* $\exists g$ *an entire function and a sequence of integers* $\{p_n\}$ *such that*

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

*where*

$$E_n(y) = \begin{cases} (1-y) & \text{if } n = 0, \\ (1-y)\exp\left(\frac{y^1}{1} + \frac{y^2}{2} + \cdots + \frac{y^n}{n}\right) & \text{if } n = 1, 2, \ldots \end{cases}$$

This is a direct generalization of the Fundamental Theorem of Algebra. It turns out that for $\sin(\pi x)$, the sequence $p_n = 1$ and the function $g(z) = \log(\pi)$ works. Here, we attempt to briefly explain why this works. We start by using the functional representation

$$\sin(\pi z) = \frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right)$$

and recognizing that the zeros are precisely the integers $n \in \mathbb{Z}$. One of the Lemmas that provides the bulk of the proof of the Factorization Theorem requires that the sum

$$\sum_{n=-\infty}^{\hat{+\infty}} \left(\frac{r}{|a_n|}\right)^{1+p_n} < +\infty$$

be finite for all $r > 0$, where the hat $\hat{}$ indicates the $n = 0$ term is removed. Since $|a_n| = n$, we see that $p_n = 1\ \forall n$ suffices, and so

$$\sin(\pi x) = z\exp(g(z)) \hat{\prod_{n=-\infty}^{\infty}} \left(1 - \frac{z}{n}\right)\exp(z/n)$$

and we can cancel the terms $\exp(z/n)\cdot\exp(-z/n)$ and combine the factors $(1 \pm z/n)$ to say

$$f(z) := \sin(\pi x) = z\exp(g(z)) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) =: \exp(g(z))zh(z)$$

for some entire function $g(z)$. Now, a useful Lemma states that for analytic functions $f_n$ and a function $f = \prod_n f_n$, we have

$$\sum_{k=1}^{\infty}\left[f_k'(z)\prod_{n\neq k} f_n(z)\right]$$

which immediately implies

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty}\frac{f_n'(z)}{f(z)}$$

This allows us to write

$$\begin{aligned}\pi\cot(\pi z) = \frac{f'(z)}{f(z)} &= \frac{g'(z)\exp(g(z))}{\exp(g(z))} + \frac{1}{z} + \sum_{n=1}^{\infty}\frac{-2z/n^2}{1 - z^2/n^2}\\ &= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2}\end{aligned}$$

and according to previous analysis, we know

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2}$$

which is based on integrating $\int_\gamma \pi(z^2 - a^2)^{-1}\cot(\pi z)\,dz$ for $a$ non-integral and where $\gamma$ is an appropriately chosen rectangle. As we enlarge $\gamma$, the integral goes to 0, and we get what we want. This means $g(z) = c$ for some $c$. Putting this back into the formula above, we have

$$\frac{\sin(\pi z)}{\pi z} = \frac{\mathrm{e}^c}{\pi}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

for all $0 < |z| < 1$. Taking $z \to 0$ tells us $\mathrm{e}^a = \pi$, and we're done!

**Remark 4.** *Notice that plugging in $z = \frac{1}{2}$ and rearranging yields the aforementioned Wallis' product for $\pi$*

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}$$

Now, let's define the Riemann zeta function and discuss some of the interesting number theoretical applications thereof.

**Definition 5.** *The Riemann zeta function is defined as the analytic continuation of the function defined by the sum of the series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \Re(s) > 1$$

This function is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. A remarkable elementary result in this field is the following.

**Theorem 6** (Euler product formula)**.** *For all $s > 1$,*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \ prime} \frac{1}{1 - p^{-s}}$$

*Sketch.* Start with the sum definition for $\zeta(s)$ and subtract off the sum $\frac{1}{2^s}\zeta(s)$:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$
$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots$$

We see that this removes all terms $\frac{1}{n^s}$ where $2 \mid n$. We repeat this process by taking the difference between

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots$$
$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \cdots$$

and we see that this removes all of the terms $\frac{1}{n^s}$ where $2 \mid n$ or $3 \mid n$ or both. Continuing ad infinitum, we have

$$\cdots\left(1 - \frac{1}{11^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1$$

and dividing by the factors on the left yields the desired result. □

**Remark 7.** *To prove a neat consequence of this formula, let's consider the case $s = 2$. We have*

$$\prod_{p \ prime}\left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.6079271016$$

*Let's think about what the product on the left hand side represents. Given two random integers $m, n$, the probability that $2 \mid m$ is $\frac{1}{2}$ (and so is $2 \mid n$) since "roughly half" of the integers are divisible by two. Likewise, the probability that $3 \mid m$ is $\frac{1}{3}$ (and same for $3 \mid n$). Thus, each term in the product is just the probability that a prime $p$ does* not *divide both $m$ and $n$. Multiplying over all primes $p$ gives us the probability that $m, n$ have no common factors, i.e. that $m$ and $n$ are* relatively prime*, or $gcd(m, n) = 1$.*

The Riemann zeta function has a non-series definition, given by

$$\zeta(s) = 2^s \pi^{s-1} \; \sin\left(\frac{\pi s}{2}\right) \; \Gamma(1-s) \; \zeta(1-s)$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

is the so-called Gamma function that analytically extends the factorial function to the complex plane. This is what allows us to think of $\zeta(s)$ as a function over the whole complex plane (except $s = 1$). Notice that $\zeta(-2k) = 0$ for any $k \in \mathbb{N}$, since the sin term vanishes; these are the so-called *trivial* zeros of the $\zeta$ function. The famous *Riemann hypothesis* asserts that

$$\Re z = \frac{1}{2} \quad \forall \zeta(z) = 0 \text{ nontrivial}$$

This problem is famously unsolved (and is worth quite a bit of fame and money), and it is generally believed to be true; in fact, "modern computer calculations have shown that the first 10 trillion zeros lie on the critical line," (according to Wikipedia) although this is certainly not a proof, and some have even pointed to Skewes' number (an astronomically huge number that served as an upper bound for some proof) as an example of phenomena occurring beyond the scope of current computing power. The bottom line is that this is closely related to the distribution of prime numbers in $\mathbb{N}$. Riemann's original explicit formula involves defining the function

$$f(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$$

where $\pi(x)$ is the number of primes less than $x$. If we can find $f(x)$, then we can recover

$$\pi(x) = f(x) - \frac{1}{2} f(x^{1/2}) - \frac{1}{3} f(x^{1/3}) - \cdots$$

Riemann's huge result is that

$$f(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) - \log(2) + \int_x^\infty \frac{dt}{t(t^2-1)\log(t)}$$

where $\mathrm{Li}(x)$ is the *logarithmic integral* function

$$\mathrm{Li}(x) = \int_0^x \frac{dx}{\log(x)}$$

and the sum $\sum_\rho$ is over the *nontrivial* zeros of $\zeta(s)$. There are a number of results that have been shown to be true after assuming the hypothesis, and there are a variety of conditions that have been shown to be equivalent to the hypothesis. From Wikipedia: "In particular the error term in the prime number theorem is closely related to the position of the zeros; for example, the supremum of real parts of the zeros is the infimum of numbers $\beta$ such that the error is $O(x^\beta)$." An interesting equivalency is that "$\zeta$ has only simple zeros on the critical line is equivalent to its derivative having no zeros on the critical line." Also from Wikipedia: "Patterson (1988) suggests that the most compelling reason for the Riemann hypothesis for most mathematicians is the hope that primes are distributed as regularly as possible." and "The proof of the Riemann hypothesis for varieties over finite fields by Deligne (1974) is possibly the single strongest theoretical reason in favor of the Riemann hypothesis."

It's also possible that Riemann initially made the conjecture with the hopes of proving the Prime Number Theorem.

**Theorem 8** (Prime Number Theorem)**.** *Let* $\pi(x) = |\{p < x\}|$*. Then*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

We also know that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0$$

so that there are "infinitely many less primes than composite integers," which gives some heuristic credence to the result above regarding the (relatively high) probability that 2 random integers are relatively prime.

This area is incredibly rich, deep, diverse and mathematically stimulating, but most of it is way behind what we have time to discuss here. It's about time that we move on and race through a bunch of other proofs of Euler's original result.

At least, while we're on the tangential topic of complex analysis, let's do a complex proof, i.e. use the calculus of residues. We'll make use of the following ideas. We will frequently use Euler's formula

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta)$$

which also allows us to define, for $z \in \mathbb{C}$,

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i} \qquad \cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}$$

**Definition 9.** *Suppose* $U \subseteq \mathbb{C}$ *is open and* $\exists a$ *such that* $f$ *is holomorphic over* $U \setminus \{a\}$*. If* $\exists g : U \to \mathbb{C}$ *holomorphic and a positive integer* $n$ *such that*

$$f(z) = \frac{g(z)}{(z-a)^n} \qquad \forall z \in U \setminus \{a\}$$

*then* $a$ *is called a* pole *of order* $n$*. A pole of order 1 is called a* simple pole.

**Definition 10.** *For a meromorphic function* $f$ *with pole* $a$*, the* residue *of* $f$ *at* $a$*, denoted by* $\text{Res}(f, a)$*, is defined to be the unique value* $R$ *such that* $f(z) - \frac{R}{z-a}$ *has an analytic antiderivative in some punctured disk* $0 < |z-a| < \delta$*. For a simple pole, we can use the formula*

$$\text{Res}(f, c) = \lim_{z\to c}(z-c)f(z).$$

*An alternative definition is to use the Laurent expansion about* $z = a$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

*If $f$ has an isolated singularity at $z = a$, then the coefficient $a_{-1} = \operatorname{Res}(f, a)$. Finally, the definition we will likely use is as follows: Suppose $f$ has a pole of order $m$ at $z = a$ and put $g(z) = (z - a)^m f(z)$; then*

$$\operatorname{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$$

**Theorem 11** (Residue theorem)**.** *Assume $f$ is analytic except for singularities at $z_j$, and that $C$ is a closed, piecewise smooth curve. Then*

$$\frac{1}{2\pi i} \int_C f(z)\, dz = \sum_{j=1}^{k} \operatorname{Res}(f, z_j)$$

Now, we're ready for proof number 2!

*Proof #2, complex analysis textbook.* Define the function

$$f(z) = \pi z^{-2} \cot(\pi z)$$

so the function has poles at $z \in \mathbb{Z}$. The pole at 0 is of order 3 because

$$z^3 f(z) = \frac{\pi z}{\sin(\pi z)} \cos(\pi z) \to 1 \text{ as } z \to 0$$

and the poles at $z = n \in \mathbb{Z}$ are simple (of order 1) because

$$(z-n) f(z) = \frac{\pi(z-n)}{\sin(\pi z)} \cdot \frac{\cos(\pi z)}{z^2} \to \pm 1 \cdot \pm \frac{1}{n^2} = \frac{1}{n^2} \text{ as } z \to n$$

So to evaluate $\operatorname{Res}(f; n)$ for $n \in \mathbb{Z}$, we simply need to apply the limit definition from above, which we just evaluated, to obtain $\frac{1}{n^2}$. It takes much more work to find $\operatorname{Res}(f; 0)$; we need to consider

$$g(z) = \pi z \cot(\pi z)$$

and evaluate

$$\frac{1}{2} \lim_{z \to 0} g''(z)$$

We find

$$\begin{aligned} g'(z) &= \pi \cot(\pi z) - \pi^2 z \csc^2(\pi z) \\ g''(z) &= -2\pi^2 \csc^2(\pi z) + 2\pi^3 z \csc^2(\pi z) \cot(\pi z) \end{aligned}$$

and then we can evaluate

$$\begin{aligned}
\lim_{z\to 0} g''(z) &= \lim_{z\to 0} -2\pi^2\csc^2(\pi z) + 2\pi^3 z\csc^2(\pi z)\cot(\pi z) \\
&= 2\pi^2 \lim_{z\to 0} \frac{\pi z\cot(\pi z) - 1}{\sin^2(\pi z)} \text{ L'H} \\
&= 2\pi^2 \lim_{z\to 0} \frac{\pi\cot(\pi z) - \pi^2 z\csc^2(\pi z)}{2\pi\sin(\pi z)\cos(\pi z)} \\
&= \pi^2 \lim_{z\to 0} \frac{\cos(\pi z)\sin(\pi z) - \pi z}{\sin^3(\pi z)\cos(\pi z)} \text{ L'H} \\
&= \pi^2 \lim_{z\to 0} \frac{\cos^2(\pi z) - \sin^2(\pi z) - 1}{3\sin^2(\pi z)\cos^2(\pi z) - \sin^4(\pi z)} \text{ L'H} \\
&= \pi^2 \lim_{z\to 0} \frac{-2\cos(\pi z)\sin(\pi z)}{3\sin(\pi z)\cos^3(\pi z) - 5\sin^3(\pi z)\cos(\pi z)} \text{ L'H} \\
&= \pi^2 \lim_{z\to 0} \frac{-2\cos^2(\pi z) + 2\sin^2(\pi z)}{3\cos^4(\pi z) + p(\sin(\pi z))} = -\frac{2\pi^2}{3}
\end{aligned}$$

and so $\mathrm{Res}(f;0) = -\frac{\pi^2}{3}$. Now, let $N \in \mathbb{N}$ and take $C_N$ to be the square with vertices $(\pm 1 \pm i)(N + \frac{1}{2})$. By the residue integral theorem, we can say

$$-\frac{\pi^2}{3} + 2\sum_{n=1}^{N} \frac{1}{n^2} = \frac{1}{2\pi i}\int_{C_N} f(z)\,dz =: I_N$$

The goal now is to show $I_N \to 0$ as $N \to \infty$. The trick is to bound the quantity $|\cot(\pi z)|^2$ on the edges of the rectangle. Consider $\pi z = x + iy$. Then we can use the identities

$$\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$$

and

$$\cos(x+iy) = \cos x\cosh y - i\sin x\sinh y$$

(which one can verify easily) to write

$$\begin{aligned}
|\cot(\pi z)|^2 &= \left|\frac{\cos(x+iy)}{\sin(x+iy)}\right|^2 = \left|\frac{\cos x\cosh y - i\sin x\sinh y}{\sin x\cosh y + i\cos x\sinh y}\right|^2 \\
&= \left|\frac{\sin x\cos x - i\sinh y\cosh y}{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}\right|^2 \\
&= \frac{\sin^2 x\cos^2 x + \sinh^2 y\cosh^2 y}{(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y)^2} \\
&= \cdots = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}
\end{aligned}$$

Thus, if $z$ lies on the vertical edges of $C_N$ then $x = \Re(\pi z) = \pm\pi\frac{2N+1}{2}$ so

$$|\cot(\pi z)|^2 = \frac{\sinh^2 y}{1 + \sinh^2 y} < 1$$

and if $z$ lies on the horizontal edges of $C_N$ then $y = \Im(\pi z) = \pm\pi\frac{2N+1}{2} =: k$ so

$$|\cot(\pi z)|^2 \le \frac{1 + \sinh^2 k}{\sinh^2 k} = \coth^2 k \le \coth^2\left(\frac{\pi}{2}\right) =: K^2 > 1$$

since coth is a monotone decreasing function on $(0,\infty)$. So far any $z$ on the rectangle $C_N$, we know

$$|f(z)| = \frac{\pi}{|z|^2}|\cot(\pi z)| \leq \pi K \cdot \left(\frac{2}{2N+1}\right)^2$$

Since the lengths of the edges of $C_N$ is $L = 4(2N+1)$, then we can say

$$|I_N| \leq \frac{1}{2\pi} \cdot 4(2N+1) \cdot \pi K \cdot \left(\frac{2}{2N+1}\right)^2 = \frac{8K}{2N+1} \to 0$$

as $n \to \infty$. Therefore,

$$-\frac{\pi^2}{3} + 2 \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} = 0 \Rightarrow \zeta(2) = \frac{\pi^2}{6}$$

□

A similar consideration of the infinite series for $\cot(\pi z)$ allows us to conclude $\zeta(2k)$ for *all* values of $k$. Specifically, we have

$$1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}}{(2k)!} \pi^{2k} B_{2k} z^{2k} = \pi z \cot(\pi z) = 1 - 2 \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right) z^{2k}$$

where $B_n$ are the *Bernoulli numbers* given by the nice linear relations

$$\begin{aligned} 2B_1 + 1 &= 0 \\ 3B_2 + 3B_1 + 1 &= 0 \\ 4B_3 + 6B_2 + 4B_1 + 1 &= 0 \\ 5B_4 + 10B_3 + 10B_2 + 5B_1 + 1 &= 0 \\ &\vdots \end{aligned}$$

This tells us

$$\zeta(2) = \frac{\pi^2}{6} \quad \zeta(4) = \frac{\pi^4}{90} \quad \zeta(6) = \frac{\pi^6}{945} \quad \zeta(8) = \frac{\pi^8}{9450} \quad \zeta(10) = \frac{\pi^1 0}{3^5 \cdot 5 \cdot 7 \cdot 11}$$

Okay, so that's enough complex analysis. Let's do some good old "calculusy" proofs.

*Proof #3, Apostol (1983).* This is taken from an article in the *Mathematical Intelligencer*. First, observe that

$$\int_0^1 \int_0^1 x^{n-1} y^{n-1}\, dx\, dy = \int_0^1 x^{n-1}\, dx \cdot \int_0^1 y^{n-1}\, dy = \frac{1}{n^2}$$

Now, we sum over all $n$ and apply the monotone convergence theorem to say

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^1 \int_0^1 \sum_{n=1}^{\infty} (xy)^{n-1}\, dx\, dy = \int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy$$

Our goal now is to evaluate this integral. There are issues with the $x$- and $y$-axes, so we transform this unit rectangle into a different rectangle in the $uv$-plane. Specifically, let

$$u = \frac{x+y}{2} \qquad v = \frac{-x+y}{2} \qquad \Longleftrightarrow \qquad x = u - v \qquad y = u + v$$

which transforms the square into the square with vertices $(0,0)$ and $(1/2,-1/2)$ and $(1,0)$ and $(1/2,1/2)$, in order. Notice that

$$\frac{1}{1-xy} = \frac{1}{1-(u^2-v^2)} = \frac{1}{1-u^2+v^2}$$

and we have the Jacobian

$$J = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad |\det J| = 2$$

Also, since we have symmetry across the $u$-axis, we can write (recognizing the arctan derivative in the integrands, from Calc II)

$$\begin{aligned}
\zeta(2) &= 2\int_0^{1/2}\int_0^u \frac{2\,dv\,du}{1-u^2+v^2} + 2\int_{1/2}^1\int_0^{1-u} \frac{2\,dv\,du}{1-u^2+v^2} \\
&= 4\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\left[\arctan\left(\frac{v}{\sqrt{1-u^2}}\right)\right]_{v=0}^{v=u} du \\
&\quad + 4\int_{1/2}^1 \frac{1}{\sqrt{1-u^2}}\left[\arctan\left(\frac{v}{\sqrt{1-u^2}}\right)\right]_{v=0}^{v=1-u} du \\
&= 4\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{u}{\sqrt{1-u^2}}\right) du \\
&\quad + 4\int_{1/2}^1 \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right) du
\end{aligned}$$

since $\arctan 0 = 0$. We now want to replace the arctan expressions above with simpler stuff. Set up the right triangle with angle $\phi$ and bases $u, \sqrt{1-u^2}$ and hypotenuse 1, and we see that

$$\arctan\left(\frac{u}{\sqrt{1-u^2}}\right) = \arcsin u$$

Next, we see that

$$\theta = \arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right) \Rightarrow \tan^2\theta = \frac{1-u}{1+u}$$

and using the identity $\tan^2\theta + 1 = \sec^2\theta$, we have

$$\sec^2\theta = 1 + \frac{1-u}{1+u} = \frac{2}{1+u} \Rightarrow u = 2\cos^2\theta - 1 = \cos(2\theta)$$

Rearranging for $\theta$ yields

$$\theta = \frac{1}{2}\arccos u = \frac{\pi}{4} - \frac{1}{2}\arcsin u$$

and this is an expression that will be helpful in the integral above. We can now straightforwardly evaluate

$$\begin{aligned}
\zeta(2) &= 4\int_0^{1/2} \frac{\arcsin u}{\sqrt{1-u^2}}\, du + 4\int_{1/2}^{1} \frac{\pi/4}{\sqrt{1-u^2}} - \frac{\frac{1}{2}\arcsin u}{\sqrt{1-u^2}}\, du \\
&= \left[2\arcsin^2 u\right]_0^{1/2} + \left[\pi \arcsin u - \arcsin^2 u\right]_{1/2}^{1} \\
&= 2\left(\frac{\pi}{6}\right)^2 - 0 + \pi\cdot\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^2 - \pi\cdot\frac{\pi}{6} + \left(\frac{\pi}{6}\right)^2 \\
&= \frac{\pi^2}{18} + \frac{\pi^2}{2} - \frac{\pi^2}{4} - \frac{\pi^2}{6} + \frac{\pi^2}{36} = \frac{\pi^2}{6}
\end{aligned}$$

□

*Proof #3b, Calabi, Beukers, Kock.* This is very similar to the previous proof. Start by observing that

$$\int_0^1\int_0^1 x^{2m}y^{2m}\, dx\, dy = \frac{1}{(2m+1)^2} \qquad n \geq 0$$

and then sum, applying the monotone convergence theorem, to get

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \int_0^1\int_0^1 \frac{1}{1-x^2y^2}\, dx\, dy$$

Make the substitution

$$x = \frac{\sin u}{\cos v} \qquad y = \frac{\sin v}{\cos u}$$

so that

$$u = \arctan\left(x\sqrt{\frac{1-y^2}{1-x^2}}\right) \qquad v = \arctan\left(\sqrt{\frac{1-x^2}{1-y^2}}\right)$$

(One can check that this actually works.) This gives us the Jacobian matrix

$$J = \begin{bmatrix} \frac{\cos u}{\cos v} & \frac{\sin u \sin v}{\cos^2 v} \\ \frac{\sin u \sin v}{\cos^2 u} & \frac{\cos v}{\cos u} \end{bmatrix}$$

and so, magically,

$$|\det J| = 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v} = 1 - x^2y^2$$

which eliminates everything in the integrand. Now, the unit square is transformed to the triangular region

$$A = \left\{(u,v) : u, v > 0 \text{ and } u + v < \frac{\pi}{2}\right\}$$

which has area $\frac{\pi^2}{8}$. It only remains to observe that

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} &= 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots \\ &= \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \cdots\right) \\ &\quad - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \cdots\right) \\ &= \zeta(2) - \frac{1}{4}\zeta(2) = \frac{3}{4}\zeta(2)\end{aligned}$$

So we've shown $\frac{3}{4}\zeta(2) = \frac{\pi^2}{8} \iff \zeta(2) = \frac{\pi^2}{6}$. □

*Proof #4, Fourier analysis textbook.* Consider the Hilbert space

$$X = L^2[0,1] = \left\{ f : [0,1] \to \mathbb{C} : \int_0^1 |f(x)|^2 \, dx < +\infty \right\}$$

with inner product

$$\langle f, g \rangle = \int_0^1 f \overline{g} \, dx$$

One can prove that the set of functions $\{e_n\}_{n \in \mathbb{Z}}$ given by

$$e_n(x) = \exp(2\pi i n x) = \cos(2\pi n x) + i \sin(2\pi n x)$$

form a *complete orthonormal* set in $X$; that is,

$$\langle e_m, e_n \rangle = \int_0^1 e_m(x) e_n(x) \, dx = 0 \quad \forall m \neq n$$

and $\|e_n\|^2 = \langle e_n, e_n \rangle = 1 \ \forall n$. Note that this uses the fact that

$$\overline{\exp(2\pi i n x)} = \exp(-2\pi i n x)$$

Basically, $\{e_n\}$ is a "basis" for $X$, in some sense. This allows us to apply *Parseval's formula*, which says that

$$\langle f, f \rangle = \|f\|^2 = \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2$$

In a Hilbert space, this formula is equivalent to being a complete orthonormal set (also called an orthonormal basis). The trick now is to apply this formula to the simple function $f(x) = x \in X$. It's easy to see that

$$\langle f, f \rangle = \int_0^1 x^2 \, dx = \frac{1}{3}$$

and

$$\langle f, e_0 \rangle = \int_0^1 x \, dx = \frac{1}{2}$$

Applying integration by parts, we can show that

$$\int_0^1 x \exp(-2\pi i n x) \, dx = -\frac{1}{2\pi i n}$$

This completes the brunt work of the proof, because now we apply Parseval's formula to say

$$\frac{1}{3} = \frac{1}{4} + \hat{\sum_{n\in\mathbb{Z}}} \frac{1}{4\pi^2 n^2} \Rightarrow \frac{1}{12} = \frac{1}{4\pi^2} 2\zeta(2) \Rightarrow \zeta(2) = \frac{\pi^2}{6}$$

□

**Remark 12.** *An essentially equivalent proof applies Parseval's formula to the function* $g = \chi_{[0,1/2]}$ *and uses the equivalent formulation for* $\frac{3}{4}\zeta(2)$ *we discussed above.*

For the next proof, we'll require DeMoivre's Formula.

**Lemma 13.** *For any* $z \in \mathbb{C}$ *and* $n \in \mathbb{Z}$*,*

$$(\cos z + i \sin z)^n = \cos(nz) + i \sin(nz)$$

*Proof.* Follow's directly from Euler's formula (see above), or can be proven easily by induction. □

*Proof #5, Apostol.* This can be found in Apostol's *Mathematical Analysis*. Note that for $0 < x < \frac{\pi}{2}$ the following inequalities hold

$$\sin x < x < \tan x \quad \Rightarrow \quad \cot^2 x < \frac{1}{x^2} < \csc^2 x = 1 + \cot^2 x$$

Let $n, N \in \mathbb{N}$ with $1 \leq n \leq N$. Then $0 < \frac{n\pi}{2N+1} < \frac{\pi}{2}$, so

$$\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{(2N+1)^2}{n^2\pi^2} < 1 + \cot^2\left(\frac{n\pi}{2N+1}\right)$$

and multiplying everything by $\frac{\pi^2}{(2N+1)^2}$ and summing from $n = 1$ to $n = N$ yields

$$\frac{\pi^2}{(2N+1)^2} A_N < \sum_{n=1}^{N} \frac{1}{n^2} < \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} A_N$$

where

$$A_N = \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

We now claim that

$$\lim_{N\to\infty} \frac{A_N}{N^2} = \frac{2}{3}$$

Before proving this claim, notice that this finishes the proof. Taking $N \to \infty$ in the two way inequality above, we obtain

$$\frac{\pi^2}{4} \cdot \frac{2}{3} \leq \zeta(2) \leq 0 + \frac{\pi^2}{4} \cdot \frac{2}{3} \Rightarrow \zeta(2) = \frac{\pi^2}{6}$$

Now, to prove the claim, take $1 \leq n \leq N$ as before and let

$$\theta = \frac{n\pi}{2N+1} \Rightarrow \sin\left((2N+1)\theta\right) = 0 \quad , \quad \sin\theta \neq 0$$

By DeMoivre's Formula,

$$\sin((2N+1)\theta) = \Im\left[(\cos\theta + i\sin\theta)^{2N+1}\right]$$

Now, if we think of actually expanding the $2N+1$ copies in the product on the right in the line above, then we can see that

$$\sin((2N+1)\theta) = \sum_{k=0}^{N}(-1)^k \binom{2N+1}{2N-2k} \cos^{2N-2k}\theta \cdot \sin^{2k+1}\theta$$

where $k=0$ corresponds to one factor of $i\sin\theta$ and the rest $\cos\theta$, $k=1$ corresponds to 3 factors of $i\sin\theta$, etc. But we know this is 0, so we can divide by the nonzero factor $\sin^{2N+1}\theta$ and distribute this into the sum to obtain

$$0 = \sum_{k=0}^{N}(-1)^k \binom{2N+1}{2N-2k} \cot^{2N-2k}\theta =: F(\cot^2\theta)$$

which is some polynomial in $\cot^2\theta$, where

$$F(x) = (2N+1)x^N - \binom{2N+1}{3}x^{N-1} + \binom{2N+1}{5}x^{N-2} - \cdots + (-1)^{N-1}\binom{2N+1}{2}x + (-1)^N$$

From above, we know that the roots $x$ such that $F(x)=0$ are precisely $\cot^2\theta$ for $1 \le n \le N$. It is a well-known result that $\sum r_n = -\frac{a_{n-1}}{a_n}$ for the roots $r_n$ of any polynomial. This tells us

$$A_N = \frac{(2N+1)!}{(2N-2)!3!(2N+1)} = \frac{N(2N-1)}{3} \Rightarrow \frac{A_N}{N^2} \to \frac{2}{3}$$

as $N \to \infty$. □

The next proof is rather slick and should be mentioned, although I won't go through the calculations at all.

*Proof #6, Fourier analysis.* If $f$ is continuous, of bounded variation on $[0,1]$, and $f(0) = f(1)$, then the Fourier series of $f$ converges to $f$ pointwise. Using the function $f(x) = x(1-x)$, we can calculate the Fourier coefficients to get

$$x(1-x) = \frac{1}{6} - \sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{\pi^2 n^2}$$

Let $x = 0$ and we get $\zeta(2) = \frac{\pi^2}{6}$ immediately. Also, letting $x = \frac{1}{2}$ gives us the interesting, and related, result

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

□

Here we list a sketch of the big ideas behind some other known proofs.

*Proof #7, Boo Rim Choe.* Taken from a note by Boo Rim Choe in the *American Mathematical Monthly* in 1987. Use the power series for arcsin:

$$\arcsin x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

which is valid for $|x| \leq 1$. Let $x = \sin t$ and apply the integral formula

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}$$

to get the $\frac{3}{4}\zeta(2)$ formulation, as before. □

*Proof #8.* The series

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$$

is uniformly convergent on $\mathbb{R}$. With some complex (i.e. imaginary numbers) calculus, we can show that

$$g(t) = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}$$

is uniformly convergent on $[\epsilon, 2\pi - \epsilon]$, by Dirichlet's test. Then for $t \in (0, 2\pi)$, we have

$$f'(t) = -g(t) = \frac{t - \pi}{2}$$

Apply the FTC from $t = 0$ to $t = \pi$ to the function $f'(t)$. □

*Proof #9, Matsuoka.* Found in *American Mathematical Monthly*, 1961. Consider

$$I_n = \int_0^{\pi/2} \cos^{2n} x \, dx$$

and

$$J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx$$

A well-known reduction formula says

$$I_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{\pi}{2}$$

and applying integration by parts to $I_n$ and simplifying yields the relation

$$\frac{\pi}{4n^2} = \frac{4^{n-1}(n-1)!^2}{(2n-2)!} J_{n-1} - \frac{4^n n!^2}{(2n)!} J_n$$

Add this from $n = 1$ to $n = N$ and then one just needs to show that

$$\lim_{N \to \infty} \frac{4^N N!^2}{(2N)!} J_N = 0$$

which is accomplished by bounding $J_N$ above by the difference $\frac{\pi^2}{4}(I_N - I_{N+1})$. □

*Proof #10.* Consider the "well-known" identity for the Fejer kernel

$$f(x) := \left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^2 = n + 2\sum_{k=1}^{n}(n-k)\cos(kx)$$

Integrate

$$\int_0^\pi x f(x)\,dx$$

to obtain some series. Let $n \to \infty$. □

*Proof #11.* Start with Gregory's formula

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

and square both sides "carefully." This is an exercise in Borwein & Borwein's *Pi and the AGM* (Wiley, 1987). □

*Proof #12.* Let $r(n)$ be the number of representations of a positive integer $n$ as a sum of four squares. Then

$$r(n) = 8 \sum_{m|n,\, 4\nmid m} m$$

Let $R(N) = \sum_{n=0}^{N} r(n)$. Then $R(N)$ is asymptotic to the volume of the 4-dimensional ball of radius $\sqrt{N}$; i.e. $R(N) \sim \frac{\pi^2}{2}N^2$. Use $r(n)$ to relate this to $\zeta$. □


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