21-801 Advanced Topics in Discrete Math: Graph Theory
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Contents

0 Introduction

1 Matchings

1.1 Matchings in Bipartite Graphs

Definition 1.1. *A* matching *is* ************

Theorem 1.2 (Hall, 1930)**.** ************

Proof. ********** □

Definition 1.3. *A k-factor of a graph G is a k-regular spanning subgraph of G.*

Note: a 1-factor is a complete matching and a 2-factor divides G into cycles.

Theorem 1.4. *Any k-regular bipartite graph has a 1-factor.*

Proof. Let $S \subseteq A$. Then $e(S, N(S)) = k|S|$ and $e(N(S), A) = k|N(S)|$. Certainly, $e(N(S), A) \geq e(S, N(S))$ so $|N(S)| \geq |S|$. Thus, Hall's condition is satisfied and so $\exists$ a matching, i.e. a 1-factor. □

Theorem 1.5 (Petersen, 1891)**.** *Every regular graph of positive, even degree has a 2-factor.*

Proof. Let G be $2k$-regular. Then we can find an Eulerian tour through G (i.e. a closed walk through vertices, of the form $v_0 v_1 \cdots v_\ell = v_0$, that visits every edge). Replace every v by (v^-, v^+) and add edge $e_i^\star = v_i v_{i+1}$ □

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1.2 Matchings in General (Simple) Graphs

For a given graph G, let $q(G)$ denote the number of odd components of G.

Theorem 1.6 (Tutte, 1947). *G has a 1-factor $\iff q(G-S) \leq |S|\ \forall S \subseteq V$ (Tutte's Condition, or TC).*

Proof. ($\Rightarrow$) If G has a 1-factor then

($\Leftarrow$) Suppose TC holds and G has *no* 1-factor. Add edges to G to form $G^\star$ such that $G^\star$ has no 1-factor but $G^\star + e$ contains a 1-factor for any possible additional edge e. Then $q(G^\star - S) \leq q(G-S) \leq |S|$ for all $S \subseteq V$. If $S = \emptyset$ then $q(G^\star) = 0$ and $|V|$ is even. Consider $U = \{v \in V : d_{G^\star}(v) = n-1\}$ where $n = |V|$. Notice $U \neq V$ otherwise $G^\star$ is a complete graph on $n = 2k$ vertices, so it would have a matching.
Claim: $G^\star - U$ is a disjoint union of complete graphs. **Proof**: Suppose not. Then $\exists x, y, z$ such that $xy, yz \in E(G^\star)$ and $xz \notin E(G^\star)$. Since $y \notin U$, $\exists$ such that $yw \notin E(G^\star)$. Let M_1 be the matching in $G^\star + xz$ and M_2 be the matching in $G^\star + yw$. Let $H = M_1 \Delta M_2$. Notice H is a disjoint union of even cycles.
Case 1: xz and yw belong to different cycles. ****
Case 2: xz and yw belong to the same cycle. ****** □

Theorem 1.7 (Petersen). *Every bridgeless cubic graph has a 1-factor.*

Proof. Pick S, arbitrary. Let C be an odd component of $G-S$. Then $e(C,S) \geq 3$. Let e be the number of edges from S to odd components. Then $3q(G-S) \leq e \leq 3|S|$ by our assumptions. Thus, TC holds and so G has a 1-factor. □

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Goal for today is to state and prove a theorem stronger than Tutte's theorem in that it implies Tutte's theorem and tells us some other stuff.

Definition 1.8. *A graph $G = (V,E)$ is* factor-critical *if $G \neq \emptyset$ and $G - v$ has a 1-factor $\forall v \in V$.*

Definition 1.9. *Let $\mathcal{C}_g$ be the components of G. A vertex set $S \subseteq V$ is called* matchhable *to $\mathcal{C}_{G-S}$ if the graph obtained by contracting components of $G-S$ to single vertices and deleting edges within S contains a matching of S.*

The following is Theorem 2.2.3 from Diestel.

Theorem 1.10. *Every graph $G = (V,E)$ contains a vertex set $S \subseteq V$ with the following two properties:*

1. *S is matchable to $\mathcal{C}_{G-S}$*
2. *Every component of $G-S$ is factor-critical*

Given such an S, the graph contains a 1-factor $\iff |S| = |\mathcal{C}_{G-S}|$.

Why does this imply Tutte's theorem? The first property of S implies $|S| \leq |\mathcal{C}_{G-S}|$ and the second condition implies $|\mathcal{C}_{G-S}| = q(G-S)$. Tutte's condition then implies $|\mathcal{C}_{G-S}| = q(G-S) \leq |S|$, so $|S| = |\mathcal{C}_{G-S}|$.

Proof. The 1-factor $\iff$ part follows from properties (1) and (2):
($\Rightarrow$) If $\exists$ 1-factor then $q(G-S) \leq |S| \leq q(G-S)$ so $|S| = |\mathcal{C}_{G-S}|$.
($\Leftarrow$) If $|S| = |\mathcal{C}_{G-S}|$ then match S to one vertex of each component in $\mathcal{C}_{G-S}$ and then use factor-criticality to find a matching in each component with one vertex removed (accounting for the matchability to S).

Now, to show existence of S, we use induction on $|G|$.
Base case: $|G| = 0$. Take $S = \emptyset$.
Indcutive step: Let G be given, $|G| > 0$ and suppose the theorem holds for graphs with fewer vertices. Conisder the sets $T \subseteq V$ where Tutte's condition fails "the worst", i.e.

$$d(T) := d_G(T) := q(G-T) - |T|$$

is a maximum. So $d(T) \geq d(\emptyset) \geq 0$. Let S be a largest such set.

Claim 1: Every component $C \in \mathcal{C}_{G-S} =: \mathcal{C}$ is odd.

Proof of Claim 1: Suppose some $C \in \mathcal{C}$ is even. Pick a vertex $c \in C$ and let $T := S \cup \{c\}$. WWTS $d(T) \geq d(S)$ to obtain a contradiction. Notice $C - \{c\}$ has odd order and so it has at least one odd component which is also a component of $G - T$. Then

$$d(T) = q(G-T) - |T| \geq q(G-S) + 1 - (|S|+1) = q(G-S) - |S| = d(S)$$

and this contradicts our assumption that S was the largest set that maximumized d. This proves Claim 1.

Claim 2: Every $C \in \mathcal{C}$ is factor-critical.

Proof of Claim 2: Suppose $\exists C \in \mathcal{C}$ and $c \in C$ such that $C' := C - \{c\}$ has no 1-factor. By the inductive hypothesis, $\exists S' \subseteq V(C')$ such that $q(C' - S') > |S'|$ (using the fact that the current theorem implies Tutte's Theorem). Notice $|C'|$ is even, so if $|S'|$ is even then $q(C' - S')$ is even (since $|C' - S'|$ is even, too); similarly, if $|S'|$ is odd then $q(C' - S')$ is odd. Thus, $q(C' - S') \geq |S'| + 2$, using the previously established inequality. Furthermore, we have two equalities involving $|T| := S \cup \{c\} \cup S'$:

$$q(G-T) = q(G-S) + q(C'-S') \quad \text{and} \quad |T| = |S| + 1 + |S'|$$

Then,

$$d(T) = q(G-T) - |T| = q(G-S) - 1 + q(C'-S') - |S| - 1 - |S'| \geq q(G-S) - |S| = d(S)$$

This proves Claim 2.

Claim 3: S is matchable to $\mathcal{C}_{G-S}$.

Proof of Claim 3: Suppose not. Then $\exists S' \subseteq S$ such that $|N_{\mathcal{C}}(S')| < |S'|$ by Hall's Theorem. Let $T = (S')^c$ and $S = S' \cup T$. So,

$$d(T) = q(G-T) - |T| \geq q(G-S) - |N_{\mathcal{C}}(S')| - |T| > q(G-S) - |S'| - |T| = q(G-S) - |S|$$

This proves Claim 3 and completes the proof. □

Let M be any matching and $k_M :=$ number of edges in M with at least 1 end in S, and let $k_G :=$ number of edges in M with both ends in $G - S$. Notice M satisfies $k_S \leq |S|$ and $k_G \leq \frac{1}{2}(|V| - |S| - |\mathcal{C}|)$. Any maximum matching satisfies these at equality.

Theorem 1.11 (Gallai-Edmonds Structure Theorem)**.** *Let $G = (V, E)$ be any graph. Let D be the set of vertices which are not covered by at least one maximal matching. Let A be the vertices in $V - D$ which are adjacent to at least 1 vertex in D. Let $C = V - D - A$. Then*

1. *The components of $G[D]$ are factor critical.*
2. *$G[C]$ has a perfect matching*
3. *The bipartite graph on $A \cup \mathcal{C}_{G[D]}$ has positive surplus viewed from A; that is, $N(S) > |S|$ for every $S \subseteq A$ ($S \neq \emptyset$).*
4. *Any maximal matching has*
   - *a near perfect matching of components of $G[D]$*
   - *perfect matchings on components of $G[C]$*
   - *matches each vertex with distinct components of $G[D]$*
5. *$|M| = \frac{1}{2}(|V| - c(G[D]) + |A|)$, where $c(\cdot)$ is the number of components.*

Proof. ****************** □

Definition 1.12. *$\mathcal{H}$ has the* Erdös-Posa property *if there $\exists f : \mathbb{N} \to \mathbb{R}$, $k \mapsto f(k)$, such that $\forall k$ either G contains k disjoint subgraphs, each isomorphic to a graph in $\mathcal{H}$, or there is a set $U \subseteq V(G)$ with $|U| \le f(k)$ such that $G - U$ has no subgraph in $\mathcal{H}$.*

Goal: prove class of all cycles has E-P property (with $f(k) \approx 4k \log k$). For the rest of today, consider

$$r_k := \log k + \log\log k + 4 \qquad s_k := \begin{cases} 4kr_k & \text{if } k \ge 2 \\ 1 & \text{if } k \le 1 \end{cases}$$

Lemma 1.13. *Let $k \in \mathbb{N}$ and let H be a cubic (3-regular) multigraph (loops and multiple edges allowed). If $|H| \ge s_k$ then H contains k disjoint cycles.*

Proof. Induction on k. **Base case**: $k \le 1$ trivial. **Inductive step**: Let $k \ge 2$ be given and let C be a shortest cycle in H. **CLaim**: $H - C$ contains a subdivision of a cubic multigraph H' with $|H'| \ge |H| - 2|C|$.

*** subdivision picture ***

Proof of claim: Let m be the number of edges between C and $H - C$. Since H is 3-regular and the average degree of C is 2, $m \le |C|$. Now, consider the following sequence of bipartitions of V, $\{V_1, V_2\}$. Start with $V_1 = V(C)$. If $H[V_2]$ has a vertex of degree ≤ 1, move it to V_1. Then the number of crossing edges decreases by ≥ 1 each time. Suppose you can do this n times, but no more. Then $\{V_1, V_2\}$ is crossed by $\le m - n$ edges. Hence $H[V_2]$ has at most $m - n$ vertices of degree < 3 and these vertices have degree $= 2$ (otherwise we moved it over to V_1). Now "suppress" the vertices of degree 2 in $H[V_2]$ (i.e. delete such a vertex v and add an edge between its neighbors). This yields a cubic graph multigraph H'. Notice

$$|H'| \ge |H| - \underbrace{|C|}_{\text{original cycle}} - \underbrace{n}_{\text{move-over provess}} - \underbrace{(m-n)}_{\text{suppress degree 2s}} \ge |H| - 2|C|$$

This proves the claim.

Now, we just have to show $|H'| \ge s_{k-1}$. Corollary 1.3.5 (in Diestel) says if $\delta(G) \ge 3$ then $g(G) \le 2\log|G|$ (where $g(\cdot)$ is the *girth*, i.e. length of shortest cycle). So $|C| \le 2 \log |H|$. Since $|H| \ge s_k \ge 6$ and $x - 4\log x$ is increasing for $x \ge 6$, we get

$$|H'| \ge |H| - 2|C| \ge |H| - 4\log|H| \ge s_k - 4\log s_k$$

To complete the proof WWTS $s_k - 4\log s_k \ge s_{k-1}$. For $k = 2$ we have $4 \cdot 2 \cdot (1 + 0 + 4) - 4\log 40 \ge 1 = s_1$. Also, notice $r_k \le 4\log k$ for $k \ge 3$ (for $k = 3$ use a calculator, and for $k \ge 4$ it's obvious). So for $k \ge 3$

$$\begin{aligned} s_k - 4\log s_k &= 4kr_k - 4\log(4kr_k) = 4(k-1)r_k + 4\log k + 4\log\log k + 16 - (8 + 4\log k + 4\log r_k) \\ &\ge s_{k-1} + 4\log\log k + 8 - 4\log(4\log k) = s_{k-1} \end{aligned}$$

□

Theorem 1.14 (Erdös, Posa, 1965)**.** *The class of all cycles has E-P property.*

Proof. Let $f(k) := s_k + k - 1$. Let k be given and G be any graph (and assume G has a cycle, otherwise it's trivial). So it has a maximal (with respect to the subgraph relation) subgraph H where all degrees in H are either 2 or 3. Let U be its set of degree 3 vertices and let $\mathcal{C}$ be the set of cycles in G that avoid U and meet H in exactly 1 vertex. Let $Z \subseteq V(H) - U$ be the set of vertices in a member of $\mathcal{C}$. For each $z \in Z$, pick one cycle $C_z \in \mathcal{C}$ and let $\mathcal{C}' = \{C_z : z \in Z\}$. The cycles in $\mathcal{C}'$ are disjoint by maximality of H (otherwise take part of cycles until first meeting point and add to H). Let $\mathcal{D}$ be the set of 2-regular components of H that avoid Z. Then $\mathcal{C}' \cup \mathcal{D}$ is a set of disjoint cycles. So if $|\mathcal{C}' \cup \mathcal{D}| \ge k$ then we're done. Otherwise, take one vertex from each $\mathcal{D}$-cycle and add it to Z to get a set X of size $\le k - 1$ which meet all cycles in $\mathcal{C}$ and all 2-regular components of H. Consider any cycle of G which avoids X. It has to meet H by maximality. It has to meet U because: it can't be all in H (otherwise in $\mathcal{D}$), it can't meet H in just one vertex (otherwise in $\mathcal{C}$), and it can't connect 2 vertices of $H - U$ with a path outside of H, so it must hit U. So every cycle in G meets $X \cup U$. We know $|X| \le k - 1$. If $|U| < s_k$ then we have $< f(k)$ vertices meeting each cycle. If $|U| \ge s_k$, suppress all degree 2 vertices in H to get a 3-regular multigraph H' with $|H'| = |U| \ge s_k$. Apply the lemma. □

1.3 Tree Packing

Let G be a given graph.

Theorem 1.15 (Menger's Theorem)**.** *If G is k-edge connected, then $\exists k$ disjoint paths between any 2 vertices in G.*

Question: How many edge-disjoint spanning trees exist in G?
Necessary condition: k-edge connectivity.
Is this condition sufficient? **No**. Consider $k = 2$ and take a cycle with 4 vertices.
Another necessary condition: for all partitions of $V(G)$ into r sets, each spanning tree has $\geq r-1$ cross-edges (edges with ends in different partitions).

Theorem 1.16 (Nash-Williams 1961, Tutte 1961)**.** *A multigraph G has k edge-disjoint spanning trees $\iff$ G has $\geq k(r-1)$ cross edges for any partition of size r.*

Corollary 1.17. *Every $2k$-edge connected multigraph G has k edge-disjoint spanning trees.*

Proof of Corollary. ***** picture $\Rightarrow G$ has $\geq \frac{1}{2}\sum_{i=1}^{r} 2k = kr \geq k(r-1)$. **Open**: is this bound sharp? □

Set up for proof of theorem: Let G be a given multigraph and $k \in \mathbb{N}$. Let $\mathcal{F}$ be the set of all k-tuples $F = (F_1, F_2, \ldots, F_k)$ where the F_is are edge-disjoint spanning forests such that $\|F\| := |E[F]| = |E[F_1] \cup \cdots \cup E[F_k]|$ is as large as possible.

If $F \in \mathcal{F}$ and $e \in E \setminus E[F]$ then $F_i + e$ for $i = 1, 2, \ldots, k$ constains a cycle. For some fixed i, take e' in this cycle ($e' \neq e$). Then setting $F_i' := F_i + e - e'$ and $F_j' := F_j$ for $j \neq i$ yields a new $F' = (F_1', \ldots, F_k')$ such that $F' \in \mathcal{F}$. We say F' is obtained from F by the *replacement* of e' with e. Note: for every path $x \ldots y \in F_i'$ $\exists! xF_iy$.

Consider the fixed k-tuple $F^0 = (F_1^0, \ldots, F_k^0) \in \mathcal{F}$. Let $\mathcal{F}^0$ be the set of all k-tuples that can be obtained from F^0 by a series of edge replacements. Let $E^0 := \bigcup_{F \in \mathcal{F}^0} (E \setminus E[F])$ and $G^0 := (V, E^0)$.

Lemma 1.18. *For any $e^0 \in E \setminus E[F^0]$ there exists $U \subseteq V(G)$ that is connected in every F_i^0 and contains the ends of e^0.*

Proof. *** "we believe the lemma" □

Proof of Theorem. ($\Leftarrow$) Induction on $|G|$. **Base case**: $|G| = 2$. Done. **Induction step**: Suppose for each partition P of $V(G)$, $\exists \geq k(|P|-1)$ cross edges. We will construct k edge-disjoint spanning trees.

Fix a k-tuple $F^0 = (F_1^0, \ldots, F_k^0) \in \mathcal{F}$. If each F_i^0 is a tree, done; otherwise,

$$\|F^0\| = \sum_{i=1}^{k} \|F_i^0\| < k(|G|-1)$$

(Recall: $\|\cdot\|$ denotes # of edges.) We have $\|G\| \geq k(|G|-1)$ by assumption, when we consider P to be single vertices. Thus, $\exists e^0 \in E \setminus E[F^0]$. By the Lemma , $\exists U \subseteq V(G)$ that is connected in each F_i^0 and contains ends of e^0. In particular, $|U| \geq 2$.

Since every partition of the contracted multigraph $G \setminus U$ induces a partition of G with the same # of cross edges, $G \setminus U$ has $\geq k(|P|-1)$ cross edges, with respect to any partition P. By induction, $G \setminus U$ has k disjoint spanning trees $T_1, \ldots, T_k$. In each T_i, replace V_U by the spanning tree $F_i^0 \cap G[U]$.

Apparently the other direction is obvious. □

We say subgraphs $G_1, \ldots, G_k$ partition G if their edge sets form a partition of $E(G)$.
Question: Into how many connected spanning subgraphs can we partition a given G?
If we can answer that question, then we can answer the **question**: Into how few acyclic spanning subgraphs can we partition G? Or, for a given k, which graphs can be partitioned into k forests?
Necessary: $\forall U \subseteq V(G)$ induces $\leq k(|U|-1)$ edges.

Theorem 1.19 (Nash-Williams 1961)**.** *A multigraph G can be partitioned into at most k forests* $\iff$ $\|G[U]\| \leq k(|U|-1)$ *for all* $u \subseteq V(G)$*.*

Proof. We will show: every k-tuple $F = (F_1, \ldots, F_k) \in \mathcal{F}$ partitions G. Suppose otherwise; then $\exists e \in E \setminus E[F]$. Use the Lemma By the Lemma, $\exists U \subseteq V$ connected in every F_i and containing ends of e. Therefore, $G[U]$ has $|U|-1$ edges in each F_i in addition to e, so $\|G[U]\| > k(|U|-1)$, a contradiction. □

1.4 Path Covering (for digraphs)

Definition 1.20. *A* path partition *in a digraph D is a family of vertex disjoint directed paths that cover all of the vertices of D. We let $\alpha(D)$ denote the maximum size of an independent set in D.*

Theorem 1.21 (Gallai-Milgram 1960)**.** *Every digraph D has a path partition with $\leq \alpha(D)$ paths.*

Proof. By induction on $|D|$. Will show: if $\mathcal{P}$ is a path partition with $|\mathcal{P}| > \alpha(D)$ then $\exists \mathcal{Q}$ with $|\mathcal{Q}| = |\mathcal{P}| - 1$ and $\text{Start}(\mathcal{Q}) \subseteq \text{Start}(\mathcal{P})$ where $\text{Start}(\mathcal{P})$ is the set of starting vertices of paths in $\mathcal{P}$.

****** picture

Let P_u be a path starting at u, for some $u \in \text{Start}(\mathcal{P})$. Since $|\mathcal{P}| > \alpha(D)$, $\exists \vec{uv}$ where $v \in \text{Start}(\mathcal{P})$. If $\text{len}(P_u) = 0$, replace P_v by uvP_v and we're done. If $\text{len}(P_u) \geq 1$, then $\exists \vec{uw} \in P_u$. Let $D' = D - u$. Notice $\alpha(D') \leq \alpha(D)$. Let $\mathcal{Q}$ be a path partition of D'. Notice $|\mathcal{Q}| = |\mathcal{P}| > \alpha(D) \geq \alpha(D')$. By induction, $\exists \mathcal{Q}$ such that $|\mathcal{Q}'| = |\mathcal{Q}| - 1$ and $\text{Start}(\mathcal{Q}') \subseteq \text{Start}(\mathcal{P}) - \{u\} + \{w\}$. □

Let μ be the size of a maximal matching in a bipartite digraph. Then the min size of a path cover is $n - 2\mu + \mu = n - \mu$ where X has n vertices. ***** picture

Corollary 1.22. *König's Theorem*

Corollary 1.23 (Dilworth 1960)**.** *In every finite poset $(P, \leq)$, max size of an antichain = min size of chain partition.*

Proof. Let e be a chain partition and A the max antichain. Certainly $|e| \geq |A|$. WWTS $|A|$ chains suffice. Use Gallai-Milgram on D with edges $\{(x, y) : x < y\}$. In this graph, antichain $\leftrightarrow$ independent set and chain cover $\leftrightarrow$ path cover. □

1.5 Connectivity

Definition 1.24. *G is* k-connected *if the minimum size of a separator is $\geq k$. The connectivity $\kappa(G) =$* $\max k$ *such that G is k-connected.*

Definition 1.25. *A* block *is a maximal connected (sub)graph with no cut-vertex.*

Examples of blocks are K_1, bridges, maximal 2-connected subgraphs, etc. We can form a natural *block graph* that is a bipartite graph with one set of vertices as the blocks and the other set as the cut vertices that blocks share.

***** picture

Proposition 1.26. *The block graph of a connected graph is a tree.*

Proof. G is connected $\Rightarrow$ the block graph is connected. Can the block graph have cycles? *No*, by the maximality of blocks. □

Proposition 1.27. *A graph is 2-connected* $\iff$ $\exists$ *sequence cycle* $= G_0, G_1, \ldots, G_n = G$ *such that* G_{i+1} *is obtained from* G_i *by adding a* G_i*-path.*

***** picture ***** "ear decomposition"

Proof. ($\Leftarrow$) Trivial, since G_i 2-connected $\Rightarrow G_{i+1}$ 2-connected.
($\Rightarrow$) Given a sequence $G_0, G_1, \dots, G_i$, suppose $G_i \neq G$. Then $\exists e \in E(G) - E(G_i)$. Let $e = xy$ with $x \in V(G_i)$. We know $G - x$ is connected so $\exists$ path from y to $z \in V(G_i)$. Adding e gives G_{i+1}. □

Theorem 1.28 (Tutte 1961)**.** *G is 3-connected $\iff \exists$ seq $G_0, G_1, \dots, G_n = G$ such that G_{i+1} has an edge $e = xy$ with $\deg(x), \deg(y) \geq 3$ and $G_i = G_{i+1}/e$.*

Lemma 1.29. *If G is 3-connected with $|V(G)| > 4$ then $\exists e \in E(G)$ such that G/e is 3-connected.*

Proof. Suppose not. Then $\forall e = xy \in E(G)$, G/e has a cut-set of ≤ 2 vertices. We know, then, that $V_{xy} \in S$, the cut-set, since $\kappa(G) \geq 3$. Let $S = \{V_{xy}, z\}$. Then $X = \{x, y, z\}$ is a cut-set of $G \Rightarrow$ every vertex in X has an edge to every component of $G - X$. Let C be the smallest component of $G - X$ over all $\{x, y, z\}$. Let $w \in N(z) \cap C$. By assumption, G/f has a cut-set of size ≤ 2, so $\exists v$ such that $\{v, z, w\}$ is a cut-set of G and each of these vertices has an edge to every component of $G - \{v, z, w\}$. Since x, y are connected, $\exists$ component D that does not contain x and y. Thus, $\emptyset \neq D \subseteq N(w) \cap V(C)$, so $D \subsetneq C$, contradiction our assumption that C was the smallest such component. ******* picture ********* □

Proof of theorem. ($\Rightarrow$) by Lemma
($\Leftarrow$) K_4 is 3-connected. Suppose G_i is 3-connected but G_{i+1} is not. Then $G_i = G_{i+1}/e$ ($e = xy$) where G_{i+1} is 2-connected. Let S be a cut-set of size ≤ 2; let C_1, C_2 be two components of $G_{i+1} - S$. Since x, y are connected, we may assume $V(C_1) \cap \{x, y\} = \emptyset$. Also, C_2 cannot contain both x and y (otherwise S is a cutset of G_i), nor can it contain any $v \notin \{x, y\}$ (otherwise V_{xy} will be disconnected from C_1 in G_i by removing ≤ 2 vertices). This is a contradiction of the degree assumption! □

Theorem 1.30 (Tutte's Wheel Theorem)**.** *Every 3-connected graph can be obtained by the following procedure:*

- *Start with K_4*
- *Given G_i pick a vertex v*
- *Split into v' and v'' and add edge $\{v', v''\}$*

Today we decide whether k-connectivity is equivalent to having k independent paths.

Definition 1.31. *Let $A, B \subseteq V(G)$. An $A - B$ path is a path $P = (u, \dots, v)$ where $P \cap A = \{u\}$ and $P \cap B = \{v\}$.*

A set S is an $A - B$ separator if there is no $A - B$ path in $G - S$.

Theorem 1.32 (Menger 19217)**.** *The minimum size of an $A - B$ separator = maximum number of disjoint $A - B$ paths.*

Proof. Let k = min size of an $A - B$ separator. Clearly #paths $\leq k$. We will construct k disjoint $A - B$ paths, by induction on $|E(G)|$.
Base: $|E(G)| = 0$. Here $|A \cap B| = k$ and k vertices form trivial paths.
Inductive: Suppose $xy = e \in E(G)$. Consider G/e. Put V_{xy} in A or B (or both) if x or y is in A or B. Suppose the max # of disjoit $A - B$ paths in G is $\leq k - 1$. Then the same holds in G/e. By induction, $\exists A - B$ separator S' of size $\leq k - 1$ and $V_{xy} \in S'$ (otherwise S' is an $A - B$ separator in G).
Now, $S = S' \setminus \{V_{xy}\} \cup \{x, y\}$ is a separator in G of size k. Consider $G' = G - e$. Note: every $A - S$ separator and every $S - B$ separator is also an $A - B$ separator; therefore, the mnimum size of an $A - S$ separator is $\geq k$. By induction, $\exists k$ disjoint paths from A to S, likewise from S to B. These paths cannot intersect outside $A \cup S \cup B$. Since $|S| = k$, combine the 2 sets of paths. Done. □

Definition 1.33. *Suppose $B \subseteq V(G)$ and $a \in V(G) \setminus B$. An $a - B$ fan is a set of paths from a to B that intersect only at a.*

Corollary 1.34 (Fan Theorem)**.** *Min # of vertices needed to separate a from B = max size of an $a - B$ fan.*

Proof. Apply Menger's Theorem to $A = N(a)$ and B. ☐

Corollary 1.35 (Local Version of Menger's Theorem)**.**

1. *If $ab \notin E(G)$, then min size of $a - b$ separator = max # internally disjoint $a - b$ paths.*
2. *If $a \neq b$, min # edges needed to separate a from b = max # edge disjoint $a - b$ paths.*

Proof.

1. Apply Menger's Theorem to $A = N(a)$ and $B = N(b)$.
2. Apply Menger's Theorem to the line graph of G: $A = E(a) := \{e \in E(G) : e \text{ is incident to } a\}$ and $B = E(b)$.

☐

Corollary 1.36 (Global Version of Menger's Theorem)**.**

1. *G is k-connected $\iff \exists k$ independent paths between any 2 vertices.*
2. *G is k-edge-connected $\iff \exists k$ edge-disjoint paths.*

Proof.

1. Done except when $ab \in E(G)$ (rest follows from Local Version 1). Suppose $ab \in E(G)$ and let $G' = G - ab$. If G' has $k - 1$ disjoint $a - b$ paths, we're done, so suppose otherwise. Then we know the max # disjoint $a - b$ paths in G' is $\leq k - 2$ and so, by Menger's Theorem, $\exists$ an $a - b$ separator S of size $\leq k - 2$. Since $|V(G)| > k$, $\exists w \notin S \cup \{a, b\}$. S is either an $a - w$ separator or a $b - w$ separator (otherwise $\exists$ an $a - b$ path not hitting S), but $S \cup \{b\}$ is an $a - w$ separator in G of size $k - 1$. This is a contradiction.
2. Follows from the Local Version 2.

☐

Definition 1.37. *A graph G is k-linked if for any two sets of size k (say, with vertices $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$) we can find disjoint paths from a_i to b_i.*

Observation: k-linked $\Rightarrow$ k-connected.
Question: If a graph is $f(k)$-connected, can this be enough to guarantee k-linked? Is this even possible? If so, for which $f(k)$ is this true?

Theorem 1.38 (Jung, Larman, Mani 1970)**.** *If a graph is 2^{10k^2}, this be enough to guarantee k-linked.*

Observation: If graph is k-connected then the average degree $\geq$ minimum degree $\geq k$.

Proposition 1.39. *If a graph has average degree d, then it has a subgraph with all degrees $> \frac{d}{2}$.*

Proof sketch, algorithmic. Algorithm for finding that subgraph: if we have any vertex of degree $\leq \frac{d}{2}$, throw it away. **Question**: Why does this stop before all vertices gone? **Answer**: As we do this, the average degree is nondecreasing (basically...). The condition $E \geq \frac{d}{2}n$ is preserved.

$$\text{New E} = \text{Old E} - \text{degree of deleted vtx} \geq \text{Old E} - \frac{d}{2} \geq \frac{d}{2}n - \frac{d}{2} = \frac{d}{2}(n-1)$$

☐

Proposition 1.40. *If all degrees in graph are $\geq \delta$ then graph has cycle of length $\geq \delta + 1$.*

Proof. Suppose you take the longest path and let v be the last vertex. All neighbors of v must fall back onto path, otherwise there's a longer path. Since there are $\geq \delta$ such neighbors, choose the furthest one from v along the path and close the path to make a cycle. This has length $\geq \delta + 1$. ☐

Corollary 1.41. *If average degree* $\geq d$ *then we have a cycle of length* $\geq \frac{d}{2} + 1$.

Definition 1.42. *A graph has a topological* K_r *minor if* $\exists r$ *branch vertices and* $\binom{r}{2}$ *vertex-disjoint paths connecting them.*

Question: What average degree, if any, is enough to guarantee existence of a topological K_r minor?

Remark 1.43. Turán's Theorem says average degree $\geq\approx \frac{r-2}{r-1}n \Rightarrow K_r$ subgraph. We hope for a bound that does not depend on n.

Lemma 1.44. *Average degree* $\geq 2^{\binom{r}{2}} \Rightarrow$ *we have a topological* $K - r$ *minor.*

Proof. Consider only $r \geq 3$. **Induction** (on m): Prove the statement: average degree $\geq 2^m$ where $m = r, r+1, \ldots, \binom{r}{2}$ then we have a topological minor with r vertices and m edges (topologically, meaning we have r branch vertices and some m vertex-disjoint paths between them).
Base case: Given average degree $d \geq 2^r$, find a topological minor with r vertices and r edges, i.e. an r-cycle. By the previous proposition/observation, we have a cycle of length $\geq 2^{r-1} + 1 \geq r$. We can turn this topologically into an r-cycle by choosing r of the vertices if the length is $> r$.
Inductive step: Assume true for $m - 1$. Given average degree $d \geq 2^m$. Would be nice if we could get a connected set U such that inside $N(u)$ the average degree is $\geq 2^{m-1}$. Then, by induction we can find a topological copy of r vertices with $m - 1$ edges. Connected back to U gives us an extra edge to make m, since U is connected. Now, we need to find such a set U.
Since the average degree $\geq 2^m$ in all of G some component of G has average degree $\geq 2^m$, so WOLOG G is connected. Pick U *maximal* such that U is connected and if U is contracted then $\frac{\text{edges}}{\text{vtxs}} \geq 2^{m-1}$. Suppose $\deg(v)$ inside $N(U)$ is $< 2^{m-1}$. Then what if we were to add v to U? U would still be connected, and after contracting $U + v$ then nw edges $\geq 2^{m-1}(\text{vtxs}) - 2^{m-1}$. This contradicts U being maximal. We know such a U exists because we can pick U to be any one vertex with high degree. □

Theorem 1.45. *If a graph is* $\left(\geq 2^{\binom{3k}{2}} + 2k\right)$*-connected then it is* k*-linked.*

Proof. Fix any vertex sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ and find disjoint paths between each a_i and b_i. Find a topological K_{3k}; notice it's still $\geq 2^{\binom{3k}{2}}$-connected. Menger's Theorem allows us to connect $3k$ branch vertices with $2k$ vertex-disjoint paths **** picture **** while minimizing the # of edges not on the topological K_{3k}. Let $c_1, \ldots, c_k$ be the unused branch vertices. **** picture **** show there can't be crossing of topological T path and Menger M path. □

Theorem 1.46 (Thomas, Wollan 2005)**.** $2k$*-connected and average degree* $\geq 10k \Rightarrow k$*-linked.*

Note: $10k$-connected implies hypotheses of theorem.

Corollary 1.47. *Average degree* $\geq 8r^2 \Rightarrow \exists$ *topological* K_r *minor.*

Proof. (Mader's Theorem, Diestel Thm 1.4.3) $\Rightarrow$ have subgraph which is $\geq r^2$-connected and has average degree $\geq 5r^2 \Rightarrow \frac{1}{2}r^2$-linked. Pick r branch vertices and $r - 1$ neighbors of each. This is r^2 vertices and can dictate links between all $\frac{1}{2}r^2$ pairs of vertices. □

Let $G = (V, E)$ with an enumeration of the edges $e_1, e_2, \ldots, e_m$. We want to define a vector space with m dimensions.

Definition 1.48. *Formally, we let* $G = (V, E)$ *be a* fixed *graph with* $|V| = n$ *and* $|E| = m$. *The* edge space $\mathcal{E}(G)$ *is the* vector space *over* $\mathbb{F}_2$ *of all functions* $f : E \to \mathbb{F}_2$ *with the usual vector addition on* $\mathbb{F}_2$ *(so this corresponds to the* symmetric difference Δ *of two subgraphs of* G*). Note: a basis for the edge space is* $\{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$.

For any subspace $\mathcal{F} \subseteq \mathcal{E}(G)$, let

$$\mathcal{F}^{\perp} = \{D \in \mathcal{E}(G) : \langle F, D \rangle = 0 \; \forall F \in \mathcal{F}\}$$

where

$$\langle F, F' \rangle = \sum_{i=1}^{m} \lambda_i \lambda'_i \qquad \text{for } F = (\lambda_1, \ldots, \lambda_m), F' = (\lambda'_1, \ldots, \lambda'_m)$$

Notice $\dim \mathcal{F} + \dim \mathcal{F}^{\perp} = m$.

Definition 1.49. *The cycle space $\mathcal{C} = \mathcal{C}(G)$ is the subspace of $\mathcal{E}(G)$ spanned by all cycles in G.*

Question: What is $\dim \mathcal{C}$? Goal: Prove $\dim \mathcal{C} = m - n + 1$ (when G is connected, otherwise we just consider each component separately).

Proposition 1.50. *The induced cycles in G generate its entire cycle space.*

Proof. By induction on the number of vertices in a given cycle. □

Proposition 1.51. *TFAE:*

1. *$F \in \mathcal{C}(G)$.*
2. *F is a disjoint union of (edges sets of) cycles.*
3. *All vertex degrees of the graph (V, F) are even.*

Proof. (1)⇒(3): Symmetric difference preserves the even parity.
(3)⇒(2): Induction on $|F|$. If $F \neq \emptyset$ then F contains a cycle $\mathcal{C}$. Remove those edges and repeat.
(2)⇒(1): By definition, disjoin union is a sum of vectors. □

Definition 1.52. *The* cut space *of G is* *****************

Proposition 1.53. *Together with $\emptyset$, the cuts in G form a subspace $\mathcal{C}^{\star}$. This space is generated by cuts of the form $E(v)$.*

Example 1.54. ******* picture

Proof. Let $\mathcal{C}^{\star}$ denote the set of all cuts in G, plus $\emptyset$. WWTS $D, D' \in \mathcal{C}^{\star} \Rightarrow D + D' \in \mathcal{C}^{\star}$. Recall $D + D' = D \Delta D'$. Set $\hat{V}_1 = (V_1 \cap V'_1) \cup (V_2 \cap V'_2)$ and $\hat{V}_2 = (V_1 \cap V'_2) \cup (V_2 \cap V'_1)$. Then $D + D'$ corresponds to all edges between $\hat{V}_1$ and $\hat{V}_2$, so it is also a cut. ("pick the diagonal", essentially)

Next, $E(V_1, V_2) = \sum_{v \in V_1} E(v)$. ********** picture □

Definition 1.55. *A* minimal *non-empty cut in G is a* bond.

Remember: minimal in the sense of containment of the sets of crossing edges.

Example 1.56. ************** pictures

Observation: A cut is a bond (in a connected graph) $\iff$ both sides of the corresponding vertex partition are conneceted induced subgraphs.

Proposition 1.57. *Every cut is a disjoint union of bonds.*

Proof. Take $D \in \mathcal{C}^{\star}$. Look at the components of V_1 and V_2 .... ************ □

Theorem 1.58. *The cycle space $\mathcal{C}$ and the cut space $\mathcal{C}^{\star}$ of any graph satisfy*

1. *$\mathcal{C} = (\mathcal{C}^{\star})^{\perp}$ and*
2. *$\mathcal{C}^{\star} = \mathcal{C}^{\perp}$*

Proof. (1) WWTS $\mathcal{C} \subseteq (\mathcal{C}^\star)^\perp$. Note that any cycle in G has an even number of edges in each cut. Also, observe that $\langle F, F' \rangle = 0 \iff F$ and F' have an even number of edges in common. So $\langle C, D \rangle = 0$ for all $D \in \mathcal{C}^\star$.
For the other direction, WWTS $F \notin \mathcal{C} \Rightarrow F \notin (\mathcal{C}^\star)^\perp$. So $\exists v \in V(F)$ such that $\deg(v)$ is odd. So then $\langle E(v), F \rangle = 1$.

(2) It suffices to show that $\mathcal{C}^\star = ((\mathcal{C}^\star)^\perp)^\perp$. [This is true, for free, assuming some knowledge of finite-dimensional vector spaces.]
First, $F \in \mathcal{C}^\star \Rightarrow \forall F' \in (\mathcal{C}^\star)^\perp$ we have $\langle F, F' \rangle = 0$. Next, $\dim \mathcal{C}^\star + \dim(\mathcal{C}^\star)^\perp = m = \dim(\mathcal{C}^\star)^\perp + \dim((\mathcal{C}^\star)^\perp)^\perp$. ☐

Definition 1.59. *Let G be a given connected graph and let T be a spanning tree of G. Let $e \in E(G) \setminus E(T)$. Then C_e is the fundamental cycle with respect to T.*

Definition 1.60. *Given G and T a spanning tree and $e \in E(T)$, then D_e is the fundamental cut.*

Theorem 1.61. *Let G be a fixed connected graph and let T be a fixed spanning tree of G. Then the corresponding fundamental cycles and cuts form a basis of $\mathcal{C}$ and $\mathcal{C}^\star$, respectively. Also,* $\dim \mathcal{C} = n - m + 1$ *and* $\dim \mathcal{C}^\star = n - 1$.

Proof. Pick $e \in E(T)$ and try to write the fundamental cut as a sum. You can't! So the set of fundamental cuts is a linearly independent subset of $\mathcal{C}^\star$, and thus $\dim \mathcal{C}^\star \geq n - 1$. Similarly, $\dim \mathcal{C} \geq m - n + 1$. Now,

$$\dim \mathcal{C}^\star + \dim \mathcal{C} = m = (n-1) + (m - n + 1) \leq \dim \mathcal{C}^\star + \dim \mathcal{C}$$

so they're all equal. ☐