The Total Variation Flow with Measure Initial Data

F. Andreu*, **V. Caselles****, **J. M. Mazón*** and **S. Moll***
(*) Dept. de Análisis Matemático, Universitat de Valencia.
(**) Dept. de Tecnologia, Universitat Pompeu-Fabra,

We consider the minimizing total variation flow in $\mathbb{R}^N$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{Du}{|Du|}\right) \qquad \text{in } Q_T =]0,T[\times\mathbb{R}^N, \tag{1}$$

coupled with the initial condition

$$u(0) = \mu, \qquad \mu \text{ being a Radon measure in } \mathbb{R}^N. \tag{2}$$

This PDE appears (in a bounded domain Ω) in the steepest descent method for minimizing the total variation, a method introduced by L. Rudin, S. Osher and E. Fatemi in the context of image denoising and reconstruction.

In previous papers thes gradient descent flow (**??**) has been studied in a bounded domain under Neumann and Dirichlet boundary conditions, obtaining existence and uniqueness results for initial data in $L^1(\Omega)$. Our aim is to continue the study of the flow (**??**) when the initial conditions are Radon measures in $\mathbb{R}^N$. We shall not consider general measures, instead we shall restrict ourselves to the case of measures

$$\mu = h + \alpha\mathcal{H}^k \llcorner S, \tag{3}$$

where $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\alpha \geq 0$, $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^N$ and S is a k-manifold in $\mathbb{R}^N$ of class $W^{3,\infty}$.

We study limit solutions obtained by weakly approximating the initial measure μ by functions in $L^1(\mathbb{R}^N)$, characterizing these limit solutions when the initial condition $\mu = h + \mu_s$ where $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $\mu_s = \alpha\mathcal{H}^k \llcorner S$, $\alpha \geq 0$, k is an integer and S is a k-dimensional manifold with bounded principal curvatures. In case $k < N-1$ we prove that the singular part of the solution does not move, it remains equal to μ_s for all $t \geq 0$ and its absolutely continuous part is the unique strong solution of the Cauchy problem associated with (**??**) with initial datum h. In particular, $u(t) = \delta_0$ when $u(0) = \delta_0$. In case $k = N-1$ we prove that the singular part of the limit solution is $(1 - \frac{2}{\alpha}t)^+\mu_s$ and we also characterize its absolutely continuous part.