21-122 - Integration and Approximation
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21-122 - Week 8, Recitation 2

Section 11.1 - Sequences

64. (a) Determine whether the sequence defined as follows is convergent or divergent.

$$a_1 = 1, \qquad a_{n+1} = 4 - a_n \text{ for } n \geq 1$$

(b) What happens if we have $a_1 = 2$?

Solution - (a) We have

$$a_1 = 1, \qquad a_2 = 4 - a - 1 = 3, \qquad a_3 = 4 - a_2 = 1, \qquad a_4 = 4 - a_3 = 3$$

So the sequence can be written as $\{1, 3, 1, 3, 1, 3, \dots\}$. This sequence is divergent.

(b) If $a_1 = 2$, then $a_2 = 4 - a_1 = 2$, $a_3 = 4 - a_2 = 2$, and clearly $a_n = 2$ for all n. Thus, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = 2$. $\square$

71. Suppose $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

Solution - Since the sequence is decreasing, it's monotonic. Since $5 \leq a_n \leq 8$ for all n, the sequence is bounded. Therefore, the sequence is convergent.

Let $L = \lim_{n\to\infty} a_n$. Since $5 \leq a_n \leq 8$ for all n, then $5 \leq L \leq 8$. In fact, since the sequence is decreasing, we know that $5 \leq L < 8$. $\square$

Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

72. $a_n = (-2)^{n+1}$
73. $a_n = \frac{1}{2n+3}$
74. $a_n = \frac{2n-3}{3n+4}$
78. $a_n = n + \frac{1}{n}$

Solution - (72) The terms alternate in sign, so the sequence is not monotonic. Also, the sequence is not bounded above or below, since as n gets large, we can find terms that are arbitrarily large (when n is odd) or arbitrarily small (when n is even).

(73) This sequence is decreasing, since as n increases, $2n + 3$ increases, so $\frac{1}{2n+3}$ decreases. The sequence is also bounded, since

$$0 < a_n \leq a_1 = \frac{1}{2(1)+3} = \frac{1}{5}$$

for all $n \geq 1$. (Note: Since the sequence is decreasing, it is bounded above by its first term.)

(74) It helps to rewrite the terms of this sequence. Write

$$a_n = \frac{2n-3}{3n+4} = \frac{\frac{2}{3}(3n)-3}{3n+4} = \frac{\frac{2}{3}(3n+4-4)-3}{3n+4} = \frac{2}{3} + \frac{\frac{2}{3}(-4)-3}{3n+4} = \frac{2}{3} - \frac{\frac{17}{3}}{3n+4}$$

As n increases, $3n+4$ increases, so $\frac{1}{3n+4}$ decreases, which means that $\frac{2}{3} - \frac{\frac{17}{3}}{3n+4}$ increases. Thus, the sequence is increasing. The sequence is bounded, since

$$-\frac{1}{7} = \frac{2(1)-3}{3(1)+4} = a_1 \leq a_n = \frac{2}{3} - \frac{\frac{17}{3}}{3n+4} \leq \frac{2}{3}$$

for all $n \geq 1$. (Note: Since the sequence is increasing, it is bounded below by its first term.)

(78) For all $n \geq 1$, we have

$$a_{n+1} - a_n = (n+1+\frac{1}{n+1}) - (n+\frac{1}{n}) = 1 + \frac{1}{n+1} - \frac{1}{n} = 1 + \frac{n-(n+1)}{n(n+1)} = 1 - \frac{1}{n(n+1)} > 0$$

Therefore, $a_{n+1} > a_n$, so the sequence is increasing. The sequence is not bounded, since for each n, $a_n \geq n$, so a_n grows arbitrarily large. $\square$

81. Show that the sequence defined by

$$a_1 = 1, \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

Solution - We proceed by induction. The statement we wish to prove is: For all $n \geq 1$, $a_n < 3$ and $a_{n+1} > a_n$.

For $n = 1$, this is easy, since

$$a_1 = 1 < 3, \qquad a_2 = 3 - \frac{1}{a_1} = 3 - 1 = 2 > a_1$$

Now suppose that $a_n < 3$ and $a_{n+1} > a_n$. We want to show that $a_{n+1} < 3$ and $a_{n+2} > a_{n+1}$. Since $a_n < 3$ and a_n is positive (because the sequence is increasing so far), we have $\frac{1}{a_n} > \frac{1}{3}$, so now

$$a_{n+1} = 3 - \frac{1}{a_n} < 3 - \frac{1}{3} < 3$$

Furthermore,

$$a_{n+2} = 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n} = a_{n+1},$$

since $a_{n+1} > a_n$.

By induction, we are done.

By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent. Write $L = \lim_{n\to\infty} a_n$. We have

$$a_{n+1} = 3 - \frac{1}{a_n} \implies L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (3 - \frac{1}{a_n}) = 3 - \frac{1}{L}$$

Rearranging, we have

$$L - 3 + \frac{1}{L} = 0 \implies L^2 - 3L + 1 = 0 \implies L = \frac{3\pm\sqrt{9-4}}{2} = \frac{3\pm\sqrt{5}}{2}$$

Using a calculator (or estimating), we see that $\frac{3-\sqrt{5}}{2} < 1$. Since $a_1 = 1$ and the sequence is increasing, the limit cannot be less than 1, so $L = \frac{3+\sqrt{5}}{2}$. $\square$