21-122 - Week 8, Recitation 1

Section 4.8

17. Use Newton's method to find all roots of the equation

$$3\cos x = x + 1$$

correct to six decimal places.
Solution - Rewrite the equation as $x + 1 - 3\cos x = 0$ and define $f(x) = x + 1 - 3\cos x$. We have $f'(x) = 1 + 3\sin x$, so our formula for root-finding is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n + 1 - 3\cos x_n}{1 + 3\sin x_n}$$

To determine the number of roots and what our initial guesses should be, let's sketch the functions $x + 1$ and $3\cos x$ on a common set of axes.

Going by our sketch, good guesses for the roots would be 1, −2, and −4. We do the calculations below.

- First root: Start with $x_1 = 1$. We have

$$x_2 \approx 0.8924380, \quad x_3 \approx 0.8894729, \quad x_4 \approx 0.8894704, \quad x_5 \approx 0.8894704$$

- Second root: Start with $x_1 = -2$. We have

$$x_2 \approx -1.8562176, \quad x_3 \approx -1.8623564, \quad x_4 \approx -1.8623649, \quad x_5 \approx -1.8623649$$

- Third root: Start with $x_1 = -4$. We have

$$x_2 \approx -3.6822814, \quad x_3 \approx -3.6389597, \quad x_4 \approx -3.6379585, \quad x_5 \approx -3.6379580$$

To six decimal places, the roots are 0.889470, −1.862365, −3.637958. □

31. Explain why Newton's method doesn't work for finding the root of the equation $x^3 - 3x + 6 = 0$ if the initial approximation is chosen to be $x_1 = 1$.
Solution - For this equation, the Newton's method formula is $x_{n+1} = x_n - \frac{x^3 - 3x + 6}{3x^2 - 3}$. Since $x_1 = 1$ is a root of the denominator, then x_2 is not defined. □

Section 11.1
15, 17. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$(15) \quad \{-3, 2, -\tfrac{4}{3}, \tfrac{8}{9}, -\tfrac{16}{27}, \dots\}, \qquad (17) \quad \{\tfrac{1}{2}, -\tfrac{4}{3}, \tfrac{9}{4}, -\tfrac{16}{5}, \tfrac{25}{6}, \dots\}$$

Solutions - (15) Each term is obtained from the previous term by multiplying by $-\frac{2}{3}$. Therefore, we have $a_n = -3(-\frac{2}{3})^{n-1}$.

(17) The numerator of the nth term is given by n^2. The denominator is given by $n + 1$. Moreover, the terms alternate in sign between positive and negative values, starting with a positive value. Therefore, write $a_n = (-1)^{n-1}\frac{n^2}{n+1}$ □

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Determine whether the sequence converges or diverges. If it converges, find the limit.

25. $a_n = \frac{3+5n^2}{n+n^2}$

29. $a_n = \tan(\frac{2n\pi}{1+8n})$

31. $a_n = \frac{n^2}{\sqrt{n^3+4n}}$

42. $a_n = \ln(n+1) - \ln(n)$

43. $a_n = \frac{\cos^2 n}{2^n}$

53. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$

Solutions - (25) Write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{\frac{3}{n^2}+5}{\frac{1}{n}+1} = \frac{0+5}{0+1} = 5$$

(29) Let's consider the expression inside $\tan(\cdot)$. We have

$$\lim_{n\to\infty} \frac{2n\pi}{1+8n} = \lim_{n\to\infty} \frac{2\pi}{\frac{1}{n}+8} = \frac{2\pi}{0+8} = \frac{\pi}{4}$$

Therefore, $\lim_{n\to\infty} a_n = \tan(\frac{\pi}{4}) = 1$.

(31) Write

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+4n}} = \lim_{n\to\infty} \frac{n^2}{n^{3/2}\sqrt{1+\frac{4}{n^2}}} = \lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{1+\frac{4}{n^2}}} = \infty,$$

since the numerator approaches infinity and the denominator approaches 1. Therefore, this sequence is divergent.

(42) It helps to rewrite $a_n = \ln(n+1) - \ln(n) = \ln(\frac{n+1}{n}) = \ln(1+\frac{1}{n})$. Now

$$\lim_{n\to\infty} (1+\frac{1}{n}) = 1+0 = 1 \implies \lim_{n\to\infty} a_n = \ln(1) = 0$$

(43) For all n, $0 \leq \cos^2 n \leq 1$. Therefore, $0 \leq \frac{\cos^2 n}{2^n} \leq \frac{1}{2^n}$. Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{2^n}$, then $\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0$.

(53) This sequence is divergent, i.e. there is no limit L such that the terms a_n can be made arbitrarily close to L. This is because we can always find larger and larger values of n such that $a_n = 0$ or such that $a_n = 1$. $\square$