21-122 - Week 2, Recitation 2

Agenda

- Review
- 7.2 - Trigonometric Integrals: Example 8, 21, 23, 25, 29, 31, 41, 43
- Return HW 1

Review

- Evaluating $\int \tan^m x \sec^n x \, dx$.
  1. If n is even, save a power of $\sec^2 x$, express the remaining factors in terms of $\tan x$, and substitute $u = \tan x$, $du = \sec^2 x \, dx$.
  2. If m is odd, save a factor of $\sec x \tan x$, express the remaining factors in terms of $\sec x$, and substitute $u = \sec x$, $du = \sec x \tan x \, dx$.
- $\int \tan x \, dx = \ln|\sec x| + C, \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C$
- More useful formulas

$$\begin{aligned}
\sin A \cos B &= \tfrac{1}{2}(\sin(A-B) + \sin(A+B)) \\
\sin A \sin B &= \tfrac{1}{2}(\cos(A-B) - \cos(A+B)) \\
\cos A \cos B &= \tfrac{1}{2}(\cos(A-B) + \cos(A+B))
\end{aligned}$$

Section 7.2 - Trigonometric Integrals

(Example 8) Find $\int \sec^3 x \, dx$.
Solution - The substitutions $u = \tan x$, $u = \sec x$ don't work, so try integration by parts. Let $u = \sec x$, $dv = \sec^2 x \, dx$, so then $du = \sec x \tan x \, dx$ and $v = \tan x$. Now

$$\begin{aligned}
\int \sec^3 x \, dx &= \tan x \sec x - \int \tan^2 x \sec x \, dx \\
&= \tan x \sec x - \int (\sec^2 x - 1) \sec x \, dx \\
&= \tan x \sec x - \int \sec^3 x \, dx + \int \sec x \, dx \\
\implies 2\int \sec^3 x \, dx &= \tan x \sec x + \int \sec x \, dx \\
\implies \int \sec^3 x \, dx &= \tfrac{1}{2}(\tan x \sec x + \ln|\sec x + \tan x|) + C
\end{aligned}$$

□

(21) Evaluate $\int \tan x \sec^3 x \, dx$.
Solution - The power of tangent is odd, so substitute $u = \sec x$, $du = \tan x \sec x \, dx$. We have

$$\int \tan x \sec^3 x \, dx = \int \sec^2 x (\tan x \sec x \, dx) = \int u^2 \, du = \tfrac{u^3}{3} + C = \tfrac{\sec^3}{3} + C$$

□

(23) Evaluate $\int \tan^2 x \, dx$.

Solution - Write

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$$

□

(25) Evaluate $\int \tan^4 x \sec^6 x\, dx$.
Solution - The power of secant is even, so substitute $u = \tan x$, $du = \sec^2 x\, dx$ and get

$$\begin{aligned}\int \tan^4 x \sec^6 x\, dx &= \int \tan^4 x (\sec^2 x)^2 (\sec^2 x\, dx) \\ &= \int u^4 (1+u^2)^2\, du \\ &= \int u^8 + 2u^6 + u^4\, du \\ &= \tfrac{u^9}{9} + \tfrac{2}{7}u^7 + \tfrac{u^5}{5} + C \\ &= \tfrac{\tan^9 x}{9} + \tfrac{2}{7}\tan^7 x + \tfrac{\tan^5 x}{5} + C\end{aligned}$$

□

(29) Evaluate $\int \tan^3 x \sec x\, dx$.
Solution - The power of tangent is odd, so substitute $u = \sec x$, $du = \tan x \sec x\, dx$ and write

$$\int \tan^3 x \sec x\, dx = \int \tan^2 x (\tan x \sec x\, dx) = \int (u^2 - 1)\, du = \tfrac{u^3}{3} - u + C = \tfrac{\sec^3 x}{3} - \sec x + C$$

□

(31) Evaluate $\int \tan^5 x\, dx$.
Solution - Here, only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$. We have

$$\begin{aligned}\int \tan^5 x\, dx &= \int \tan^3 x (\sec^2 x - 1)\, dx \\ &= \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx \\ &= \int \tan^3 x \sec^2 x\, dx - \int \tan x (\sec^2 x - 1)\, dx \\ &= \int \tan^3 x \sec^2 x\, dx - \int \tan x \sec^2 x\, dx + \int \tan x\, dx \\ &= \tfrac{\tan^4 x}{4} - \tfrac{\tan^2 x}{2} + \ln|\sec x| + C\end{aligned}$$

where the first two integrals were evaluated by substituting $u = \tan x$, $du = \sec^2 x\, dx$. □

(41) Evaluate $\int \sin 8x \cos 5x\, dx$.
Solution - Note that $\sin 8x \cos 5x = \frac{1}{2}(\sin(8x - 5x) + \sin(8x + 5x)) = \frac{1}{2}(\sin 3x + \sin 13x)$. Then

$$\int \sin 8x \cos 5x\, dx = \frac{1}{2}\int \sin 3x + \sin 13x\, dx = -\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$$

□

(43) Evaluate $\int \sin 5\theta \sin \theta\, d\theta$.
Solution - Note that $\sin 5\theta \sin\theta = \frac{1}{2}(\cos(5\theta - \theta) - \cos(5\theta + \theta)) = \frac{1}{2}(\cos 4\theta - \cos 6\theta)$. Then

$$\int \sin 5\theta \sin\theta\, d\theta = \frac{1}{2}\int (\cos 4\theta - \cos 6\theta)\, d\theta = \frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C$$

□