21-122 - Integration and Approximation
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21-122 - Week 14, Recitation 1

Agenda

- Reminder: Concavity
- Section 10.2 - Calculus With Parametric Curves: 11, 26, 41, 63 (time permitting, 31)

Section 10.2

11. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

$$x = t^2 + 1, \qquad y = t^2 + t$$

Solution - When $\frac{dx}{dt} \neq 0$, we have

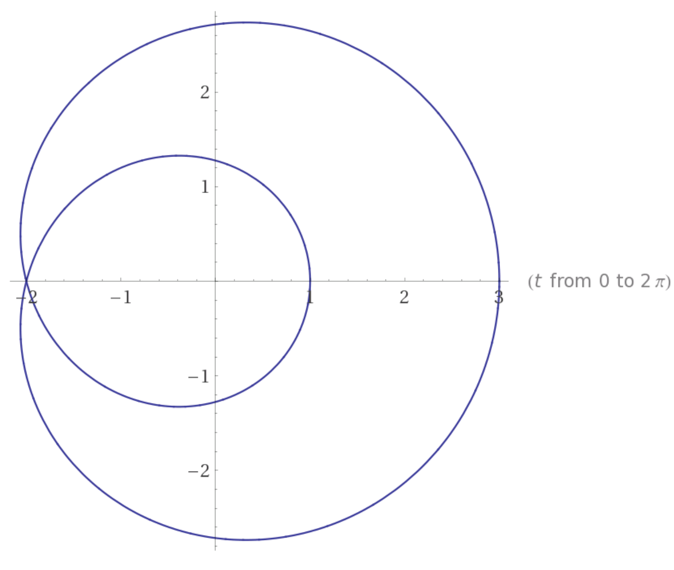
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$$

Now

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{-\frac{1}{2t^2}}{2t} = -\frac{1}{4t^3}$$

This curve is concave upward for $t < 0$. □

26. Graph the curve $x = \cos t + 2\cos 2t$, $y = \sin t + 2\sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.
Solutions -



This plot was produced by WolframAlpha. From the plot, the curve crosses itself at $(-2, 0)$. First, let's find the values of t corresponding to this point. Consider the equations

$$x = \cos t + 2\cos 2t = -2, \qquad y = \sin t + 2\sin 2t = 0, \qquad 0 \leq t \leq 2\pi$$

We focus on the second equation. We have

$$\sin t + 2\sin 2t = \sin t + 4\sin t\cos t = \sin t(1+4\cos t) = 0 \implies \sin t = 0 \text{ or } \cos t = -\tfrac{1}{4}$$

The equation $\sin t = 0$ corresponds to $t = 0$ or $t = \pi$, neither of which satisfy the first equation above. Therefore, the point $(-2, 0)$ must correspond to $\cos t = -\frac{1}{4}$, which yields t-values $t = \cos^{-1}(-\frac{1}{4})$ and $t = 2\pi - \cos^{-1}(-\frac{1}{4})$.

To find the equation of the tangents, we need $\frac{dy}{dx}$, which is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t + 4\cos 2t}{-\sin t - 4\sin 2t} = -\frac{\cos t + 4(2\cos^2 t - 1)}{\sin t + 8\sin t\cos t}$$

When $t = \cos^{-1}(-\frac{1}{4})$, $\cos t = -\frac{1}{4}$ and $\sin t = \sqrt{1-(\frac{1}{4})^2} = \frac{\sqrt{15}}{4}$. When $t = 2\pi - \cos^{-1}(\frac{1}{4})$, $\sin t = -\frac{\sqrt{15}}{4}$. Substituting these values into the formula for $\frac{dy}{dx}$, we see that the tangent lines have slopes

$$-\frac{-\frac{1}{4}+4(2\cdot(-\frac{1}{4})^2-1)}{\pm\frac{\sqrt{15}}{4}+8(\pm\frac{\sqrt{15}}{4})(-\frac{1}{4})} = \pm\frac{-1+16(\frac{1}{8}-1)}{\sqrt{15}-2\sqrt{15}} = \pm\frac{-1+2-16}{-\sqrt{15}} = \pm\sqrt{15}$$

The equations of the tangent lines are $y = \pm\sqrt{15}(x+2)$. □

41. Find the exact length of the curve.

$$x = 1 + 3t^2, \qquad y = 4 + 2t^3, \qquad 0 \le t \le 1$$

Solution

$$\begin{aligned} L &= \int_0^1 \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\, dt \\ &= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt \\ &= \int_0^1 6t\sqrt{1+t^2}\, dt \\ &= \int_1^2 3\sqrt{u}\, du \qquad (u = 1+t^2,\ du = 2t\, dt) \\ &= 2u^{3/2}\Big|_{u=1}^{2} \\ &= 2(2^{3/2} - 1) \end{aligned}$$

□

63. Find the exact area of the surface obtained by rotating the given curve about the x-axis.

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta, \qquad 0 \le \theta \le \tfrac{\pi}{2}$$

Solution

$$
\begin{aligned}
S &= \int_0^{\pi/2} 2\pi y\sqrt{(\tfrac{dx}{d\theta})^2 + (\tfrac{dy}{d\theta})^2}\ d\theta \\
&= 2\pi a\int_0^{\pi/2} \sin^3\theta\sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\ d\theta \\
&= 2\pi a\int_0^{\pi/2} \sin^3\theta\sqrt{9a^2\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\ d\theta \\
&= 6\pi a^2\int_0^{\pi/2} \sin^3\theta(\cos\theta\sin\theta)\ d\theta \\
&= 6\pi a^2\int_0^{1} u^4\ du \qquad (u = \sin\theta,\ du = \cos\theta\ d\theta) \\
&= 6\pi a^2(\tfrac{u^5}{5})\Big|_{u=0}^{1} \\
&= \tfrac{6}{5}\pi a^2
\end{aligned}
$$

□