21-122 - Week 10, Recitation 2

Agenda

- 11.4: 1, 5, 7, 17, 31
- 11.5: 3, 7, 11, 17, 23, 32

Section 11.4 - The Comparison Tests

1. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be convergent.

   (a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?

   (b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

Solutions - (a) We can't say anything about whether $\sum a_n$ converges or diverges. However, if $\sum a_n$ *does* converge, then we know $\sum a_n > \sum b_n$.

(b) We know $\sum a_n$ converges and that $\sum a_n < \sum b_n$. □

5, 7, 17, 31. Determine whether the series converges or diverges.

$$(5)\ \sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}, \qquad (7)\ \sum_{n=1}^{\infty} \frac{9^n}{3+10^n}, \qquad (17)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}, \qquad (31)\ \sum_{n=1}^{\infty} \sin(\tfrac{1}{n})$$

Solutions - (5) For all n, we have $\frac{n+1}{n\sqrt{n}} > \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (this is a p-series, $p = \frac{1}{2}$), then by the Comparison Test the original series diverges.

(7) For all $n \geq 1$, we have $\frac{9^n}{3+10^n} < \frac{9^n}{10^n} = (\frac{9}{10})^n$. The geometric series $\sum_{n=1}^{\infty} (\frac{9}{10})^n$ converges, so by the Comparison Test the original series converges.

(17) We can do this two ways. First, a solution that uses the Comparison Test. I'd like to compare $\frac{1}{\sqrt{n^2+1}}$ to something like $\frac{1}{n}$. Write

$$\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{n^2+2n+1}} = \frac{1}{\sqrt{(n+1)^2}} = \frac{1}{n+1}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges, then by the Comparison Test, so does the original series.

Another (perhaps easier) solution uses the Limit Comparison Test. Let's compare $\frac{1}{\sqrt{n^2+1}}$ with $\frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = 1$$

which is finite and positive. As above, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then so does the original series.

(31) Recall from Calculus I that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, so as $x \to 0$, we have $\sin(x) \approx x$. In light of this fact, let's compare $\sin(\frac{1}{n})$ and $\frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \overset{(x=\frac{1}{n})}{=} \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$$

which is finite and positive. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then by the Limit Comparison Test so does the original series. $\square$

More Problems

10, 12, 15, 29, 32
Determine whether the series converges or diverges.

10. $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$

12. $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

15. $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

29. $\sum_{n=1}^{\infty} \frac{1}{n!}$

32. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Solutions For (10), we can write $0 \leq \frac{k \sin^2 k}{1+k^3} \leq \frac{k}{1+k^3} \leq \frac{k}{k^3} = \frac{1}{k^2}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, then by the Comparison Test so does the given series.

For (12), rewrite the terms as

$$\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} = \frac{k^3}{k^5} \frac{(2-\frac{1}{k})(1-\frac{1}{k^2})}{(1+\frac{1}{k})(1+\frac{4}{k^2})^2} = \frac{1}{k^2} \frac{(2-\frac{1}{k})(1-\frac{1}{k^2})}{(1+\frac{1}{k})(1+\frac{4}{k^2})^2}$$

This suggests using the Limit Comparison Test with $\frac{1}{k^2}$. We have

$$\lim_{k\to\infty} \frac{\frac{1}{k^2} \frac{(2-\frac{1}{k})(1-\frac{1}{k^2})}{(1+\frac{1}{k})(1+\frac{4}{k^2})^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{(2-\frac{1}{k})(1-\frac{1}{k^2})}{(1+\frac{1}{k})(1+\frac{4}{k^2})^2} = \frac{(2)(1)}{(1)(1)^2} = 2$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does the original series.

For (15), let's compare with $(\frac{4}{3})^n$. We have

$$\lim_{n\to\infty} \frac{\frac{4^{n+1}}{3^n-2}}{(\frac{4}{3})^n} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \frac{3^n}{3^n-2} = \lim_{n\to\infty} 4 \frac{1}{1-\frac{2}{3^n}} = 4$$

Since $\sum_{n=1}^{\infty} (\frac{4}{3})^n$ diverges (geometric with $r > 1$), then so does the given series.

For (29), note that $\frac{1}{n!} = \frac{1}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} \leq \frac{1}{n(n-1)}$ for $n \geq 2$. Now

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \implies \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} (\frac{1}{n-1} - \frac{1}{n}) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots = 1$$

By the Comparison Test, the given series converges.

Finally, for (32), let's compare with $\frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^{1+1/n}}} = \lim_{n\to\infty} \frac{n^{1+1/n}}{n} = \lim_{n\to\infty} n^{1/n}$$

To find this limit, let's consider $\lim_{x\to\infty} x^{1/x}$ and use L'Hopital's Rule. We have

$$\lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Therefore, $\lim_{n\to\infty} n^{1/n} = \lim_{x\to\infty} x^{1/x} = e^0 = 1$. By the Limit Comparison Test, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then so does the given series. □