21-122 - Week 10, Recitation 1

Agenda

- 11.2: 32, 36, 40, 45, 60

Section 11.2 - Series

32, 36, 40. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$(32)\ \sum_{n=1}^{\infty} \frac{1+3^n}{2^n}, \qquad (36)\ \sum_{n=1}^{\infty} \frac{1}{1+(\frac{2}{3})^n}, \qquad (40)\ \sum_{n=1}^{\infty} (\frac{3}{5^n} + \frac{2}{n})$$

Solutions - (32) Denote $a_n = \frac{1+3^n}{2^n}$. Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{1}{2^n} + (\frac{3}{2})^n) = \infty,$$

so $\sum\limits_{n=1}^{\infty} \frac{1+3^n}{2^n}$ does not converge.

(36) Denote $a_n = \frac{1}{1+(\frac{2}{3})^n}$. We have $\lim\limits_{n\to\infty} a_n = \frac{1}{1+0} = 1 \neq 0$, so $\sum\limits_{n=1}^{\infty} \frac{1}{1+(\frac{2}{3})^n}$ does not converge.

(40) The series $\sum\limits_{n=1}^{\infty} \frac{3}{5^n}$ is convergent (this is a geometric series with $r = \frac{1}{5}$). However, the series $\sum\limits_{n=1}^{\infty} \frac{2}{n}$ is divergent, since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is known to be divergent. □

45. Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{3}{n(n+3)}$ is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.

Solution - To get a telescoping sum, we use partial fractions. Write $\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$. Solving for A and B, we have

$$3 = A(n+3) + Bn = (A+B)n + 3A \implies A = 1, B = -1$$

Thus, $a_n = \frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}$. Now

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &= (\tfrac{1}{1} - \tfrac{1}{4}) + (\tfrac{1}{2} - \tfrac{1}{5}) + (\tfrac{1}{3} - \tfrac{1}{6}) + (\tfrac{1}{4} - \tfrac{1}{7}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+3}) \\ &= \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{n+1} - \tfrac{1}{n+2} - \tfrac{1}{n+3} \\ &= \tfrac{11}{6} - \tfrac{1}{n+1} - \tfrac{1}{n+2} - \tfrac{1}{n+3} \end{aligned}$$

Now $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} (\frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}) = \frac{11}{6}$. □

60. Find the values of x for which the series $\sum\limits_{n=0}^{\infty} (-4)^n (x-5)^n$ converges. Find the sum of the series for those values of x.

Solutions - We can rewrite the terms as follows: $a_n = (-4)^n(x-5)^n = (-4(x-5))^n$. Now this is a geometric series with $r = -4(x-5)$. This series will converge provided that $|r| < 1$. Write this as

$$|-4(x-5)| < 1 \iff 4|x-5| < 1 \iff |x-5| < \tfrac{1}{4} \iff 5 - \tfrac{1}{4} < x < 5 + \tfrac{1}{4} \iff \tfrac{19}{4} < x < \tfrac{21}{4}$$

Thus, the series converges precisely when $\frac{19}{4} < x < \frac{21}{4}$. In this case, the sum of the series is

$$\sum_{n=0}^{\infty} (-4)^n (x-5)^n = \tfrac{1}{1-(-4)(x-5)} = \tfrac{1}{4x-19}$$

□

Section 11.3 - The Integral Test and Estimates of Sums

7, 11, 17, 21.

7. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is convergent or divergent.

Solution - Consider $f(x) = \frac{x}{x^2+1}$. This function is certainly continuous and positive for $x > 0$. Is it ultimately decreasing? We have

$$f'(x) = \tfrac{1\cdot(x^2+1)-2x(x)}{(x^2+1)^2} = \tfrac{1-x^2}{(x^2+1)^2}$$

Now $f'(x) < 0$ when $x > 1$, so $f(x)$ is decreasing when $x > 1$. We apply the Integral Test.

$$\int_1^{\infty} f(x)\, dx = \int_1^{\infty} \tfrac{x}{x^2+1}\, dx = \lim_{t\to\infty} \tfrac{1}{2}\ln(x^2+1)\Big|_{x=1}^{t} = \lim_{t\to\infty} \tfrac{1}{2}(\ln(t^2+1) - \ln(2)) = \infty$$

The integral is divergent, thus the sum is divergent. □

11, 17, 21. Determine whether the series is convergent or divergent.

$$(11)\ 1 + \tfrac{1}{8} + \tfrac{1}{27} + \tfrac{1}{64} + \tfrac{1}{125} + \cdots, \qquad (17)\ \sum_{n=1}^{\infty} \tfrac{1}{n^2+4} \qquad (21)\ \sum_{n=2}^{\infty} \tfrac{1}{n \ln n}$$

Solutions - (11) Rewrite this series as $\sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p-series with $p > 3$ (see page 716 of text), so it is convergent.

(17) The function $f(x) = \frac{1}{x^2+4}$ is positive, continuous, and increasing for $x \geq 1$. Write

$$\int_1^{\infty} \tfrac{1}{x^2+4}\, dx = \int_1^{\infty} \tfrac{1}{4}\tfrac{1}{(\frac{x}{2})^2+1}\, dx = \lim_{t\to\infty} \tfrac{1}{2}\arctan(\tfrac{x}{2})\Big|_{x=1}^{t} = \tfrac{1}{2}(\tfrac{\pi}{2} - \arctan(\tfrac{1}{2}))$$

The integral is convergent, so by the Integral Test, the series is convergent.

(21) The function $f(x) = \frac{1}{x \ln x}$ is continuous and positive for $x > 1$. Moreover, it is decreasing because x and $\ln x$ are increasing for $x > 1$. Since $f(x)$ is not defined at $x = 1$, we consider the integral from 2 to infinity. Write

$$\int_2^{\infty} \tfrac{1}{x \ln x}\, dx \overset{(u=\ln x)}{=} \int_{\ln 2}^{\infty} \tfrac{1}{u}\, du$$

The latter integral is known to be divergent, so the series is divergent. □