21-122 - Week 1, Recitation 1

Agenda

- Introduction
- Announcements: HW 1
- Review
- 5.5: Examples 2, 3, 4, 8
- 5.5: 12, 13, 23, 35, 45, 59, 64, 69 (time permitting: 21, 25)
- Questionnaire

Review

- The Substitution Rule, (a.k.a. u-substitution or change of variables formula): Allows us to simplify complicated integrals by making the substitution

$$u = g(x), \quad du = g'(x)\,dx$$

- How do we apply this rule?
- Idea: Choose u to be a function of x in the integrand whose differential also occurs in the integrand (except possibly for a constant factor)
- For *definite integrals*, we must also change the limits of integration.

$$\int_a^b (\cdots)dx \overset{u=g(x)}{\rightarrow} \int_{g(a)}^{g(b)} (\cdots)du$$

Section 5.5

We'll start with some examples of indefinite integrals solved via substitution

Example 2: Evaluate $\int \sqrt{2x+1}\,dx$.
Solution - Let $u = 2x+1$, so $du = 2\,dx$, or $dx = \frac{1}{2}du$. By the Substitution Rule,

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int \sqrt{u}\cdot \tfrac{1}{2}du \\ &= \tfrac{1}{2}\int u^{1/2}\,du \\ &= \tfrac{1}{2}\tfrac{u^{3/2}}{3/2} + C \\ &= \tfrac{1}{2}u^{3/2} + C \\ &= \tfrac{1}{2}(2x+1)^{3/2} + C\end{aligned}$$

□

Example 3: Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Solution - Let $u = 1 - 4x^2$. Then $du = -8x\ dx$, or $x\ dx = -\frac{1}{8}du$. By the Substitution Rule,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\ dx &= \int \frac{1}{\sqrt{u}} \cdot (-\tfrac{1}{8}du) \\ &= -\tfrac{1}{8}\int u^{-1/2}\ du \\ &= -\tfrac{1}{8}(2u^{1/2}) + C \\ &= -\tfrac{1}{4}\sqrt{1-4x^2} + C\end{aligned}$$

□

Example 4: Calculate $\int e^{5x}\ dx$.
Solution - Let $u = 5x$, so $du = 5\ dx$, or $dx = \frac{1}{5}du$. Now

$$\int e^{5x}\ dx = \int e^u \cdot \tfrac{1}{5}du = \tfrac{1}{5}\int e^u\ du = \tfrac{1}{5}e^u + C = \tfrac{1}{5}e^{5x} + C$$

□

Now we'll do a definite integral

Example 8: Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.
Solution - Substitute $u = 3 - 5x$, so $du = -5\ dx$, or $dx = -\frac{1}{5}\ du$. Note, when $x = 1$, $u = -2$. When $x = 2$, $u = -7$. Now

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\tfrac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\tfrac{1}{5}[-\tfrac{1}{u}]\Big|_{u=-2}^{-7} = \tfrac{1}{5}\tfrac{1}{u}\Big|_{u=-2}^{-7} = \tfrac{1}{5}(-\tfrac{1}{7} + \tfrac{1}{2}) = \tfrac{1}{14}$$

□

Now we'll do some problems for this section. The substitutions will be less obvious, but the same idea applies. Look for a function that appears in the integrand along with its differential
12. Evaluate $\int \sec^2 2\theta\ d\theta$.
Solution - Substitute $u = 2\theta$, so $du = 2\ d\theta$ or $d\theta = \frac{1}{2}\ du$. Now

$$\int \sec^2 2\theta\ d\theta = \tfrac{1}{2}\int \sec^2 u\ du = \tfrac{1}{2}\tan u + C = \tfrac{1}{2}\tan 2\theta + C$$

□

13. Evaluate $\int \frac{dx}{5-3x}$.
Solution - Substitute $u = 5 - 3x$, so $du = -3\ dx$, or $dx = -\frac{1}{3}\ du$. Now

$$\int \frac{dx}{5-3x} = -\tfrac{1}{3}\int \frac{du}{u} = -\tfrac{1}{3}\ln|u| + C = -\tfrac{1}{3}\ln|5-3x| + C$$

□

23. Evaluate $\int \sec^2\theta\tan^3\theta\ d\theta$.
Solution - Substitute $u = \tan\theta$, so $du = \sec^2\theta\ d\theta$. Now

$$\int \sec^2\theta\tan^3\theta\ d\theta = \int u^3\ du = \tfrac{u^4}{4} + C = \tfrac{\tan^4\theta}{4} + C$$

□

35. Evaluate $\int \sqrt{\cot x} \csc^2 x \, dx$.
Solution - Substitute $u = \cot x$, so $du = -\csc^2 x \, dx$ or $\csc^2 2x \, dx = -du$. Now

$$\int \sqrt{\cot x} \csc^2 x \, dx = -\int \sqrt{u} \, du = -\tfrac{2}{3}u^{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C$$

□

45. Evaluate $\int \frac{1+x}{1+x^2} \, dx$.
Solution - We can split this up as $\int \frac{1}{1+x^2} \, dx + \int \frac{x}{1+x^2} \, dx$. The first integral is just $\tan^{-1} x + C$. For the second, substitute $u = 1 + x^2$, so $du = 2x \, dx$, or $x \, dx = \frac{1}{2} du$. Now

$$\int \tfrac{x}{1+x^2} \, dx = \tfrac{1}{2} \int \tfrac{1}{u} \, du = \tfrac{1}{2} \ln |u| + C = \tfrac{1}{2} \ln(1 + x^2) + C$$

Thus,

$$\int \tfrac{1+x}{1+x^2} \, dx = \tan^{-1} x + \tfrac{1}{2} \ln(1 + x^2) + C$$

□

Now some definite integrals, remember to change limits
59. Evaluate $\int_1^2 \frac{e^{1/x}}{x^2} \, dx$.
Solution - Substitute $u = 1/x$, so $du = -\frac{1}{x^2} \, dx$, or $\frac{1}{x^2} \, dx = -du$. Now

$$\int_1^2 \tfrac{e^{1/x}}{x^2} \, dx = -\int_1^{1/2} e^u \, du = -e^u \Big|_{u=1}^{1/2} = e - e^{1/2}$$

□

64. Evaluate $\int_0^a x\sqrt{a^2 - x^2} \, dx$.
Solution - Substitute $u = a^2 - x^2$, so $du = -2x \, dx$, or $x \, dx = -\frac{1}{2} du$. Now

$$\int_0^a x\sqrt{a^2 - x^2} \, dx = -\tfrac{1}{2} \int_{a^2}^0 \sqrt{u} \, du = \tfrac{1}{2} \int_0^{a^2} \sqrt{u} \, du = \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} \Big|_{u=0}^{a^2} = \tfrac{1}{3}(a^2)^{3/2} = \tfrac{1}{3} a^3,$$

assuming $a \geq 0$. □

69. Evaluate $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.
Solution - Substitute $u = \ln x$, so $du = \frac{1}{x} \, dx$. Now

$$\int_e^{e^4} \tfrac{dx}{x\sqrt{\ln x}} = \int_1^4 \tfrac{du}{\sqrt{u}} = 2\sqrt{u} \Big|_{u=1}^{4} = 2(2 - 1) = 2$$

□

TIME PERMITTING

21. Evaluate $\int \frac{(\ln x)^2}{x} \, dx$.
Solution - Substitute $u = \ln x$, so $du = \frac{1}{x} \, dx$. Now

$$\int \tfrac{(\ln x)^2}{x} \, dx = \int u^2 \, du = \tfrac{u^3}{3} + C = \tfrac{(\ln x)^3}{3} + C$$

□

25. Evaluate $\int e^x \sqrt{1 + e^x} \, dx$.
Solution - Substitute $u = 1 + e^x$, so $du = e^x \, dx$. Now

$$\int e^x \sqrt{1 + e^x} \, dx = \int \sqrt{u} \, du = \tfrac{2}{3} u^{3/2} + C = \tfrac{2}{3}(1 + e^x)^{3/2} + C$$

□