Inventory Control

§1. - Introduction

We are mainly concerned here with the control of the inventory (stock) levels in an organization.

We shall develop models for a range of different situations and use a variety of techniques for their solution. The number of situations studied can only be a sample of possible problem structures and there is no all powerful "inventory management system" which can be generally applied.

Why keep Inventories?

On the face of it inventories represent idle resources and are wasteful. Money tied up in stocks could be invested elsewhere. They are, nevertheless, necessary to uncouple inputs and outputs, and usually arise from essentially "batch" inputs - either from buying in dozens to sell singly, or from naturally arising production processes. Some common example of inventories are:

1) Stocks of spare parts for cars, locomotives or any industrial machinery
2) Stocks of fuel at power stations, petrol stations
3) Stocks of semi-finished goods waiting for next stage in a production line
4) Stocks of blood in a blood bank
5) Money kept for withdrawals by banks, building societies etc.

Costs Involved

(a) Inventory Systems

(i) basic item cost, e.g. cost of purchase from supplier: often constant, but may be effected by quantity discount structures, and in dynamic situations, may change with time.

(ii) The cost of making an order excluding cost (i), e.g. cost of paper work involved: ranges from very small costs to substantial ones in some special cases.

(iii) Inventory holding costs: includes capital insurance, buildings etc. usually taken as proportional to the value of inventory held.

(iv) Stock out Costs, i.e. cost of being out of stock when an item is demanded: this is generally difficult to assess especially with regard to customer satisfaction. In some cases back orders can be 'backlogged' i.e. filled later on, while in others, non-fulfilled demands represent 'lost sales'.

Types of Inventory System

The two most common types of inventory system are:

(a) Lot Size - Re-order point systems

Orders for fixed quantities are placed when the stock level falls to a preset figure. This requires a constant monitoring of stock levels, every time an item is taken from stock.

(b) Cyclical Review Models

Stocks of a particular item are examined at fixed intervals of time and an amount is ordered depending on the stock level at that time. Often the rule is to order up to a predetermined level i.e. if stock level is s and one orders up to level L, then one orders L-s

Lead Time

There is usually a delay between making an order, and the time when the order actually arrives. This time is called the lead time. It may be constant, dependent on order size, or even a stochastic variable.

The questions we shall concern ourselves with are:

(1) When to order?

(2) How much to order?

§2 Deterministic Single Product Models

In this section we examine 3 models of an inventory system. They are necessarily very simple but illustrate some of the ideas involved.

Model 1

The demand for the product is constant and λ per period. There is a zero lead time and no stockouts are allowed. At constant intervals of time T we make an order for Q items of stock. The problem is to choose T and Q so as to minimise the average cost per period of running the system.

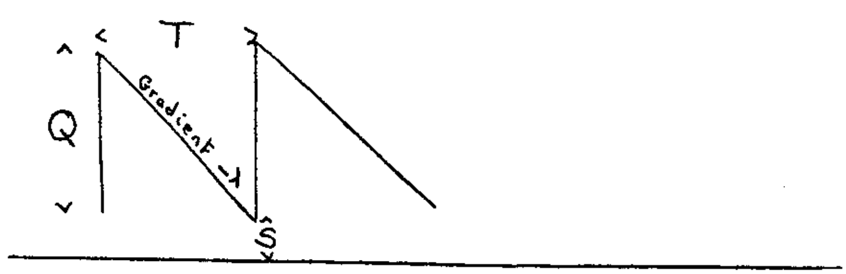
Notation

A = the fixed cost associated with making an order (a(ii))

C = unit cost per item.

I = inventory carrying charges. If one unit of stock is kept for t periods then the inventory charge is It units.

Let $S \geq 0$ be the stock level when each order arrives. The stock level will have a pattern as shown in the diagram below



From the diagram we see that $Q = \lambda T$ and so only one of these variables can be chosen independently.

Ordering Cost

The actual cost of the items bought or produced is λC per period on average regardless of the inventory policy and will not affect the choice of Q. It is therefore ignored.

The average number of orders per period will be $1/T = \lambda/Q$. Therefore the average fixed cost of making orders is

$$A\lambda/Q$$

Holding Cost.

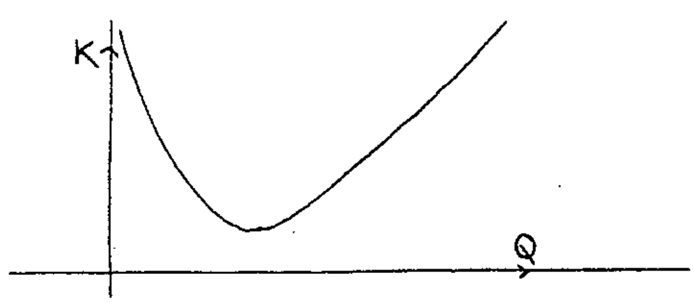
The average stock level is $\frac{1}{2}(S+(S+Q))=S+Q/2$.

$\therefore$ the average holding cost per period is

$I(S+Q/2)$

Therefore the average total variable cost per period K is given by

2.1 $K = A\lambda/Q + I(Q/2+S)$



K is to minimised for Q and $S \geq 0$. Now for given Q we minimise K by taking $S = 0$. To find the optimal Q we differentiate 2.1 (with $S = 0$).

Now $\frac{dK}{dQ} = - A\lambda/Q^2 + I/2$

Putting $\frac{dK}{dQ} = 0$ gives the optimal Q_w where

2.2 $\quad Q_w = (2\lambda A/I)^{\frac{1}{2}}$

The right hand side of 2.2 is sometimes referred to as the Wilson Lot Size Formula. Using this we see that the optimal time interval T_w and the minimum cost K_w are given by

$$T_w = (2A/\lambda I)^{\frac{1}{2}}$$

$$K_w = (2\lambda AI)^{\frac{1}{2}}$$

Discrete Case

Suppose that one can only order discrete quantities q, $2q, \ldots$ and that Q_w is not an exact multiple of q. In this case we simply compare the costs for $Q = [Q_w/q]$ and $Q = [Q_w/q] + q$ and take the smaller.

Ex

$A = 10$, $\lambda = 100$, $I = .1$ and $q = 10$

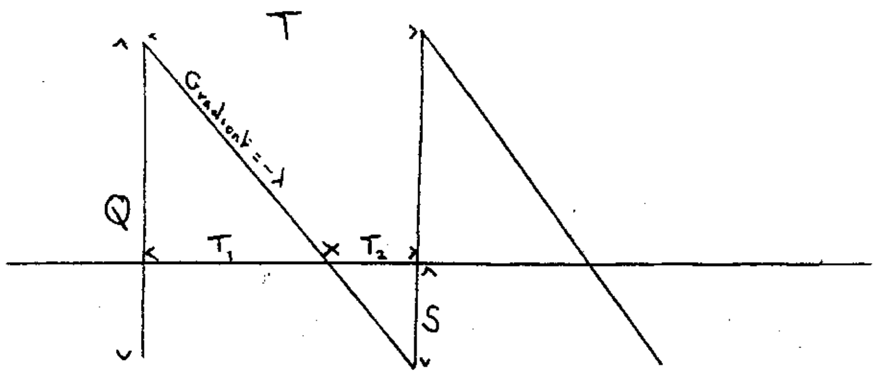
then $Q_w = 141.4$ and $K_w = 14.141$

Also $K(140) = 14.143$ and $K(150) = 14.167$

Therefore the optimal value of Q in the discrete case is $Q = 140$.

Model 2

This model is the same as for model 1 except that items out of stock can be back-ordered and supplied when goods come into stock. The cost of each item back-ordered is πt where t is the length of time the demand remains unfilled. The inventory level will follow the pattern shown below



S is the number of back orders when the order Q arrives. The objective is to choose Q and S so that the average cost per period is minimised.

The following relations are apparent from the diagram

$$Q = \lambda T$$

$$S = \lambda T_2$$

$$Q-S = \lambda T_1 \quad .$$

<u>Ordering Cost</u>

The number of orders per period is again $1/T = \lambda/Q$ and so the variable ordering cost is

$$\lambda A/Q$$

<u>Holding Cost</u>

The proportion of time that there are goods in stock is T_1/T. During this time the average inventory level is $(Q-S)/2$ and so the average holding cost per period is

$$I(Q-S)T_1/2T$$

But from above we have that $T_1/T = (Q-S)/Q$ and so this cost is in fact

$$I(Q-S)^2/2Q$$

Back-Order Cost

The proportion of time that the system is out of stock is T_2/T. Because of the given form of back-order cost we may work out the average back order cost in a similar way to that of the holding cost. During the time the system is out of stock the average amount back ordered is $S/2$. Therefore the average back-order cost is

$$\begin{aligned} &\pi ST_2/2T \\ = \ &\pi S^2/2Q \qquad \text{from above} \end{aligned}$$

The total variable annual cost K is then

$$K = \lambda A/Q + I(Q-S)^2/2Q + \pi S^2/2Q$$

In order to minimise K we solve the equations

$$\frac{\partial K}{\partial Q} = \frac{\partial K}{\partial S} = 0$$

yielding

2.3 $$-\lambda A/Q^2 - I(Q-S)^2/2Q^2 + I(Q-S) - \pi S^2/2Q^2 = 0$$

and

2.4 $$-I(Q-S)/Q + \pi S/Q = 0$$

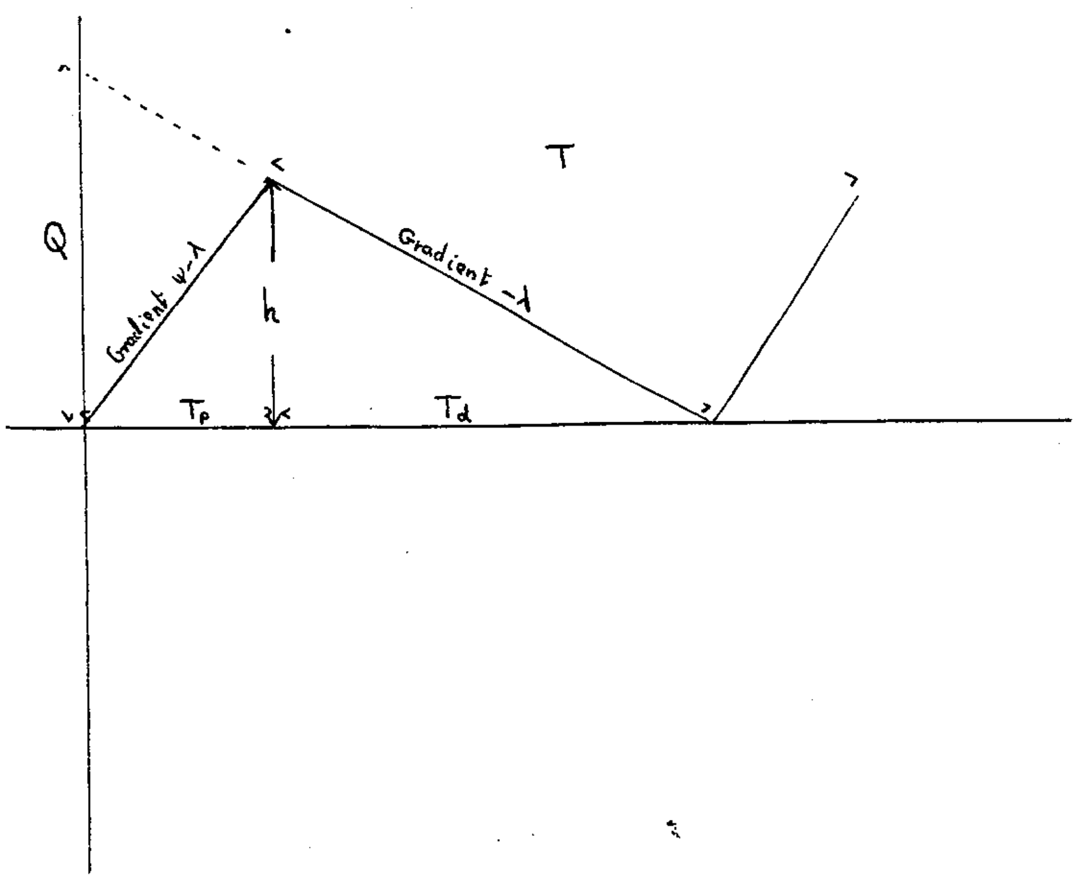
These can be solved by using 2.4 to express Q in terms of S, substitute in 2.3 to give an equation in S and then solving giving

$$\begin{aligned} S &= (2\lambda AI/\pi(\pi+I))^{\frac{1}{2}} \\ Q &= Q_w((\pi+I)/\pi)^{\frac{1}{2}} \\ K &= K_w(\pi/(\pi+I))^{\frac{1}{2}} \end{aligned}$$

Comparing with model 1 we see that the extra freedom of allowing back-orders enables us to reduce total cost. We can obtain the results for model 1 by putting $\pi = \infty$.

Model 3

In the previous models we assumed that the orders arrived instantaneously in a single lot size of Q units. We now consider a situation in which having made an order the items arrive in a continuous stream at a rate $\psi > \lambda$ per period. The inventory level assuming no back ordering will then have the pattern below:



During period T_p the stock level increases at a rate $\psi-\lambda$. During period T_d the stock level decreases at a rate λ.

If the length of time between orders is T and a total amount Q is produced at a time then we must have $Q = \lambda T$ otherwise stock levels will be zero again at the end of each cycle.

<u>Ordering Cost</u>

The average number of orders per period is again $1/T = \lambda/Q$ and so the average ordering cost is

$$A\lambda/Q$$

<u>Holding Cost</u>

If h is the maximum stock level during a cycle then the average stock is h/2. Now h is the amount of stock built up during T_p and so

$$h = (\psi - \lambda)\ T_p$$

We also have that $Q = \psi\ T_p$ as the order is produced in time T_p. Substituting in the above gives

$$h = (1 - \lambda/\psi)\ Q$$

∴ the average inventory cost per period is

$$\tfrac{1}{2}\ I(1-\lambda/\psi)\ Q$$

and the average total variable cost is

$$K = \lambda A/Q + \tfrac{1}{2}\ IQ(1-\lambda/\psi)$$

Solving the equation $\frac{dK}{dQ} = 0$ gives the optimal batch quantity as

$$Q = Q_w(\psi/(\psi-\lambda))^{\frac{1}{2}}$$

with minimal cost

$$K = K_w(\psi-\lambda)/\psi)^{\frac{1}{2}}$$

Thus total cost has decreased as against model 1 due to decrease in inventory costs. Note that taking $\psi = \infty$ gives model 1 again.

Fixed Lead Time

If the lead time τ between making an order and receiving it is a fixed constraint then its effect on the above 3 models and indeed any model is limited. We need only alter our ordering policy so that if in a certain model with zero lead time we make orders at time t_1, t_2,... then in the corresponding model with fixed lead time τ we make orders at time $t_1-\tau$, $t_2-\tau$,... and then in other respects i.e. costs and inventory levels etc., the two models will be the same.

§3 Deterministic Multi-Item Problems

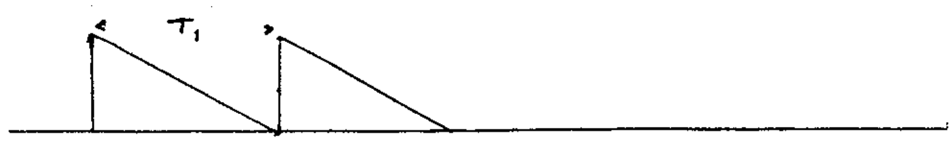
We consider here some problems when more than one type of item is being stored and when the inventory costs of the items interact. If there is no interaction we can consider each item separately.

Models 4 and 5 of this section are generalisations of model 1 although we could equally well have generalised model 2 or model 3. Model 6 however is specifically a generalisation of model 3.

Model 4

Assume that there are n distinct types of item whose individual characteristics are that of model 1. When an order is made it is possible to order more than one type of item but the fixed cost of ordering A is unaffected by the number of different types of goods ordered.

Suppose then that there is a demand λ_j and holding cost I_j for item j. Let the optimal policy be to order item j at period T_j.



We show that $T_1 = T_2 = \ldots = T_n$. For let $T = T_k = \min_j(T_j)$ Then if $T_j > T$ for some j by increasing the frequency of ordering of j to that of k and ensuring that item j is ordered at the same time as k we

(i) eliminate any extra ordering costs for j over those of k

(ii) decrease the average stock level of j.

Thus $T_j > T$ implies non-optimality and the result follows. The problem then is to find the optimal value for T.

Ordering Cost

The average number of orders per period is 1/T and so the average ordering cost per period is

$$A/T$$

Holding Cost

Let Q_j be the amount of item j ordered at a time. Then since orders are made at intervals of time T we must have

$$Q_j = \lambda_j T \tag{3.1}$$

Now the average inventory for item j is $Q_j/2$ and so the average inventory cost per period is $I_j Q_j/2$. Using 3.1 we see therefore that the average total inventory cost is

$$(\tfrac{1}{2} \sum_j \lambda_j I_j)T$$

and so the average total cost per period K is given by

$$K = A/T + (\tfrac{1}{2} \sum_j \lambda_j I_j)T$$

The optimum value for T is obtained by solving $\frac{dK}{dT} = 0$ which gives

$$T = (2A/(\sum_j \lambda_j I_j))^{\frac{1}{2}}$$

The optimal lot sizes Q_j and minimal cost K can now be calculated.

Model 5

In the previous model the items were able to share a facility (ordering) without increase of cost. Thus the total cost was smaller than if all the items were ordered separately. In some situations items will make conflicting demands on resources e.g. if there is a shortage of storage space then calculating the optimal lot size for each item separately may lead to too great a demand for storage space. We again assume that there are n items whose individual characteristics are those of model 1 but that the lot-sizes Q_j have to satisfy the constraint

3.2 $$\Sigma f_j Q_j \le f$$

For given batch sizes $Q_1, \ldots Q_n$ we may evaluate the average total cost by summing individual costs as calculated in model 1. This gives

3.3 $$K = \sum_j (\lambda_j A_j / Q_j + I_j Q_j / 2)$$

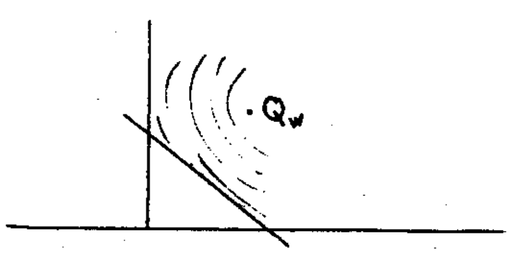
The problem is therefore to minimise K subject to 3.2 and we can proceed as follows:

(i) we first find the minimum of K ignoring constraint 3.2. Now to minimise K overall we can simply minimise each term in the summation 3.3 separately. This will naturally lead to the Wilson Lot-Sizes

$$Q_{jw} = (2\lambda A_j / I_j)^{\frac{1}{2}}$$

If these values satisfy the constraint 3.2 then we have solved the problem, otherwise

(2)



in these circumstances we may assume that constraint 3.2 is <u>active</u> at the optimum $(Q^*_1, \ldots Q^*_n)$ i.e.

$$\Sigma f_j Q^*_j = f$$

(this follows because we may express

$$K(Q) = K(Q^*) + \sum_j \frac{\partial K}{\partial Q_j}(Q_j - Q^*_j) + \ldots$$

and if $\Sigma f_j\ Q^*_j < f$ then $(Q_1 - \theta\ \frac{\partial K}{\partial Q_1}, \ldots Q_n - \theta\ \frac{\partial K}{\partial Q_n})$ is both feasible and better than Q^* for small enough $\theta > 0$ unless $\frac{\partial K}{\partial Q_j} = 0$ for $j = 1,..n$ but this would mean $Q_j = Q_{jw}$ which we have already discounted).

So we now tackle the problem

3.4 minimise K

subject to

3.5 $F = \Sigma\ f_j Q_j - f = 0$

(Strictly speaking $Q_j \geq 0$ is also a constraint but the solution to 3.4 always gives $Q_j > 0$ so we can ignore it in this case).

Problem 3.4 is in a form suitable for application of the classical Lagrange Multiplier Technique.

This would state in our case that there exists θ such that at the optimum for 3.4

$$\frac{\partial K}{\partial Q_j} = \theta\ \frac{\partial F}{\partial Q_j} \text{ for } j = 1, \ldots n \tag{3.6}$$

(a non rigourous argument for this is that if 3.6 does not hold at the optimum Q^* then there exists q such that $q.\nabla F = 0$ and $q.\nabla K < 0$

at Q^* and so for small $\theta > 0$, $Q^* + \theta q$ is both feasible and better than Q^*).

On differentiation we see that 3.6 states

$$-\lambda_j A_j / Q^2_j + I_j/2 = \theta f_j \qquad \text{for } j = 1,\ldots n$$

or

$$(3.7)\quad Q_j = (2\lambda A_j/(I_j - 2\theta f_j))^{\frac{1}{2}} \qquad \text{for } j = 1,\ldots n$$

Now the value for θ can be calculated by substituting 3.7 into F = 0 giving

$$(3.8)\quad \sum_j f_j (2\lambda_j A_j/(I_j - 2\theta f_j))^{\frac{1}{2}} = f$$

In general 3.8 may be solved using any numerical technique e.g. Newton Raphson or Bisection (Exercise:find a range for θ)

<u>Example</u>

A company manufactures 3 types of washing machine with characteristics as below:

Washing Machine	Price(£)	Sales/Week
1	80	100
2	120	80
3	150	50

The inventory charge per week is worked out as 1% of the price of the item. The fixed costs A_j for making an order are all the same at £100. The company at present manufactures the machines in batches calculated through the Wilson Lot Size formula. There is concern however at the amount of money tied up in stock under this policy and it has been decided to cut the average value of stock held by 20%. How can this be done at minimum increase in total cost of ordering and inventory?

In this case constraint 3.2 will be of the form

3.9 $$\Sigma\ C_j\ Q_j \leq C$$

where C_j is the cost of each item. Also we will have $I_j = IC_j$ for some I - in the example .01. We see then that the optimal Q_j as given by 3.7 are

$$Q_j = (2\lambda_j A_j/(IC_j - 2\theta C_j))$$

$$= (2\lambda A_j/IC_j)^{\frac{1}{2}}(1-2\theta I^{-1})^{-\frac{1}{2}}$$

$$= \alpha\ Q_{jw} \qquad \text{where } \alpha = (1-2\theta I^{-1})^{-\frac{1}{2}}$$

Thus instead of calculating θ we can calculate α and since $\Sigma C_j Q_j$ is to be cut by 20% we see that $\alpha = 0.8$. Thus the optimal lot sizes are

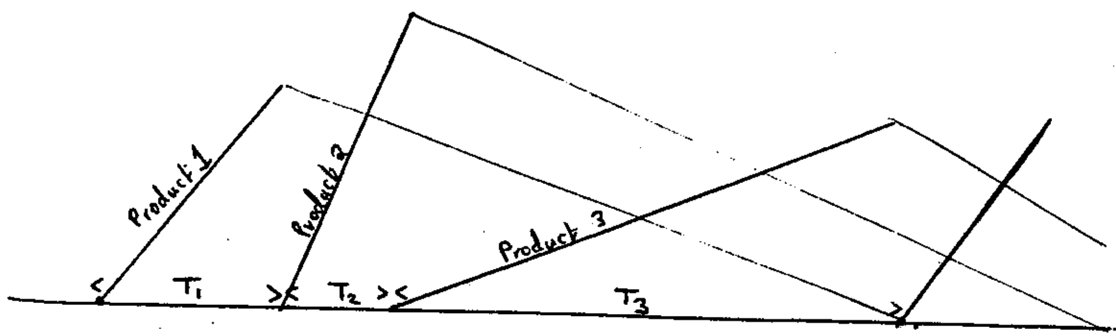
$$Q_1 = 40 \qquad Q_2 \simeq 29 \qquad Q_3 \simeq 6.5$$

The value of θ can be calculated from $\alpha = (1-2\theta I^{-1})^{-\frac{1}{2}}$ giving $\theta = -\ .0028$. This value has a similar significance to that of the shadow prices of linear programming (§6) i.e. for a small increase δC in the right hand side of 3.9 the saving in cost will be $.0028\ \delta C$.

Model 6

Suppose now that a factory manufactures n products with the characteristics of model 3 i.e. constant demand λ_j and rate of production ψ_j for product j. The products all share the same manufacturing facility and so cannot be manufactured simultaneously.

Instead they must be produced in series in batches i.e. first a batch of product j_1, then a batch of product j_2,... and so on.



This sequence of batches of products $j_1,\ldots j_n$ is to be repeated indefinitely at intervals of time T. Note that it is possible for there to be some idle time between a batch of product j_n being completed and a new batch of j_1 being started due to excess production capacity. Instead of a fixed set up cost for each product we have a change-over cost C_{ij} if a batch of product j is produced immediately after product i. The problem is to find the ordering of batches and the lengths of time interval T so that set up costs and inventory costs are minimised. Fortunately the selection of sequence and calculation of T can be done separately i.e. first find optimal sequence and then calculate T.

Finding an optimal sequence is non-trivial if the number of jobs is not small. It is in fact what is called a travelling salesman problem and the general solution of such problems is discussed elsewhere. For the example given below we simply evaluate the total change-over cost of all possible sequence.

Assume therefore that we have found the best sequence and that the total change-over cost is A.

Now let $Q_1,\ldots Q_n$ be the batch sizes of each product produced in each production cycle of length T and let $T_1, \ldots T_n$ be the time spent in each cycle producing these batches. Then we have

$$Q_j = \lambda_j T$$

and

$$Q_j = \psi_j \, T_j$$

for reasons as given in model 3.

We therefore have

3.10 $\quad T_j = \rho_j T \qquad$ where $\rho_j = \lambda_j / \psi_j$

Summing 3.10 for $j = 1, \ldots n$ we have that

$$\sum_j T_j = T \, \Sigma \, \rho_j$$

and since $\sum_j T_j \leq T$ we must have $\Sigma \rho_j \leq 1$ else it is not possible to produce the products as described above. If $\Sigma \rho_j < 1$ then there is time to spare and so another product could possibly be fitted in. Assuming then that this production method is feasible we calculate the average total change-over and inventory costs. Now the inventory levels of each individual product will be as in model 3 and so the average inventory cost for product j will be

$$I_j (1 - \lambda_j / \psi_j) Q_j / 2$$

$$= \quad I_j \, \lambda_j (1 - \rho_j) T/2$$

Therefore the average total cost is K where

3.11 $\quad K = A/T + BT \qquad$ where $B = \frac{1}{2} \sum_j I_j \lambda_j (1 - \rho_j)$

The optimal value for T is then found from $\frac{dK}{dT} = 0$ i.e.

$$T = (A/B)^{\frac{1}{2}}$$

and the values K, T_j and Q_j can then be evaluated.

Example

Suppose there are 4 products with data

Product	λ	ψ	I
1	200	600	2
2	100	400	1
3	80	400	3
4	50	250	2

Changeover costs C_{ij} are given by the matrix

-	20	30	40
30	-	20	50
40	30	-	10
30	20	40	-

The possible production sequences are (arbitrarily choose 1 as beginning of cycle)

Cycle				Changeover Cost
1	2	3	4	20 + 20 + 10 + 30 = 80
1	2	4	3	20 + 50 + 40 + 40 =150
1	3	2	4	30 + 30 + 50 + 30 =140
1	3	4	2	30 + 10 + 20 + 30 = 90
1	4	2	3	40 + 10 + 20 + 40 =110
1	4	3	2	40 + 40 + 30 + 30 =140

Thus the optimal sequence is 1-2-3-4 giving A = 80. We note that $\Sigma\rho_j$ = 59/60 and so a schedule is possible. Direct calculation gives

$$B = 133.3 + 37.5 + 96 + 40 = 406.8$$

Therefore the optimal value of $T \approx .44$.

From this we then calculate

j	1	2	3	4
T_j	.15	.11	.09	.09
Q_j	90	44	36	22

$$K = 2(AB)^{\frac{1}{2}} = 361$$

The models discussed in §2 and §3 are obviously much simplified ignoring the major factor of stochastic variation. In spite of this however the batch sizes as calculated do find a use in some practical models.

Inventory Problems

1) A firm manufactures and sells a single product. The demand for the product is 150 units/year. The value of the stock for calculating inventory costs is £10 per unit. The set up cost for a run of the product is £100. The penalty cost is considered to be £5 per unit per year out of stock. The inventory carrying charge is £1 per unit per period held.

Find the optimal re-order policy if

a. Production time is negligible and no stock-outs are allowed.

b. Production time is negligible and stock-outs can be back-ordered.

c. Production is at a rate of 400 units per year and no stock-outs are allowed.

2) A firm manufactures and sells 3 different products, production time being negligible. Data on costs and demands are given in the table below. The maximum average value of inventory that can be held at any one time is £4000. Find the optimal re-order policy under these circumstances if the inventory charge is 10% of the items value per period.

Product	Demand/year	Value/item	Set up cost
1	2000	10	50
2	1500	15	40
3	3000	20	60

3) Assuming the demand and cost data for Q2 find the optimal production policy if

(a) Production rates for products 1,2,3 are 6000, 5000, 12000 items per period respectively.

(b) Changeover costs are given by

i \ j	1	2	3
1	-	150	200
2	100	-	250
3	200	150	-

(4) If the different items can be ordered at the same time at a cost of 50 and other parameters are as in Q2 find the optimum order policy,

(5) Extend model 3 to take account of the possibility of having back-orders as in model 2.

(6) In an inventory system the cost of having q units in inventory for t units of time is Bqt^3. Demand is uniform at a constant rate of λ units per unit time. No stockouts are allowed. The fixed order cost is A. Solve the system

Model 5 - Fixed order level

It may be that it is unrealistic to assume that each additional order means an additional cost. If we consider a situation in which the sales department can deal with up to a certain number of orders per period, then it is reasonable to assume that the cost of orders up to this level remains constant. Suppose then that n items are stocked by the firm and that the parameters I, C_j, λ_j are as in the previous example. Suppose also that up to L orders per period can be dealt with. Then if x_j is the number of orders for items made per period we have the constraint

$$(1.52) \qquad \sum_{j=1}^{n} x_j \leq L$$

For simplicity let us assume that goods arrive in batches and no stock-outs are allowed. Then we only need to calculate the inventory cost per period. The amount of items ordered at a time Q must be such that all demand λ_j in that period is met.

$$Q_j = \lambda_j / x_j$$

The average stock level of items will be $\frac{1}{2}Q_j$ and so the total (inventory) cost per period is

$$(1.53) \qquad K = \sum_{j=1}^{n} \tfrac{1}{2} I C_j \lambda_j / x_j$$

The problem we must solve is that of minimising 1.53 subject to 1.52. Now one can see that no optimal solution to this problem will give a strict inequality in 1.52 because we would be able to obtain a cheaper solution by increasing x_1 for example. So we can assume equality in 1.52 and we can use a Lagrange multiplier θ as in the previous model. Define

$$J = K + \theta(\Sigma x_j - L)$$

then we have $\frac{\partial J}{\partial x_j} = 0$ in an optimal solution.

This becomes on differentiation

$$-\tfrac{1}{2} IC_j\lambda_j/x_j^2 + \theta = 0$$

(1.54) $$x_j = (IC_j\lambda_j/2\theta)^{\frac{1}{2}}$$

and we choose θ so as to make

$$\Sigma(IC_j\lambda_j/2\theta)^{\frac{1}{2}} = L$$

We take the following example with 5 different items

j	C_j	λ_j	$(C_j\lambda_j)^{\frac{1}{2}}$
1	25	50	35.03
2	5	100	22.36
3	30	150	67.08
4	2	300	24.24
5	1	600	24.24

The inventory charge is ½% giving I = .05 and the maximum number of orders per period L is 30. We have calculated the quantities $(C_j\lambda_j)^{\frac{1}{2}}$ because as can be seen from 1.54, the optimal x_j are proportional to them, and consequently we have

1.55 $x_j = L(C_j\lambda_j)^{\frac{1}{2}}/\Sigma\ (C_j\lambda_j)^{\frac{1}{2}}$

From the table we have $\Sigma(C_j\lambda_j)^{\frac{1}{2}}=173$ and therefore

$x_1 = 6.1,\ x_2 = 3.9,\ x_3 = 11.6,\ x_4 = 4.2,\ x_5 = 4.2$

If we want an integer solution, we can start by rounding the optimal solution above giving

$x_1^* = 6,\ x_2^* = 4,\ x_3^* = 12,\ x_4^* = 4,\ x_5^* = 4$

We then try to improve this solution by seeing if the total inventory cost can be decreased by increasing the number of orders for one item by one and simultaneously decreasing the number of orders of another item by one.

j	α_j	β_j
1	20.80	14.90
2	20.80	12.50
3	17.10	14.40
4	25.00	15.00
5	25.00	15.00

α_j is the increase in the average stock value of item j if one less of item j is ordered per period.

β_j is the decrease in the average stock value of item j if one more of item j is ordered per period.

We can see from this table that any such change will increase the average total value of the stock held and consequently we have found the optimal integer solution.

While it is true in general that there is some way of finding the optimal integer solution by rounding the optimal 'continuous' solution, not all roundings will do. For example if $x_1 = .25$ then rounding to $x_1 = 0$ will not be feasible.