§17. Two Person Zero Sum Games

We discuss here an application of linear programming to the theory of games. This theory is an attempt to provide an analysis of situations involving conflict and competition.

Game 1: there are two players A and B and to play the game they each choose a number 1, 2, 3 or 4 without the other's knowledge and then they both simultaneously announce their numbers. If A calls i and B calls j then B pays A an amount a_{ij} – the payoff – given in the matrix below. (If $a_{ij} < 0$ this is equivalent to A paying B $-a_{ij}$).

$$
\begin{array}{cc}
 & B \\
A & \begin{bmatrix} 2 & 4 & 2 & 1 \\ -2 & 5 & 1 & -1 \\ 1 & -5 & 3 & 0 \\ 6 & 2 & -3 & -2 \end{bmatrix}
\end{array}
$$

This is a two person zero sum game, zero sum because the algebraic sum of the players' winnings is always zero.

Game 2 (Penalty Kicks)

Suppose A and B play the following game of Soccer. A plays in goal and B takes penalty kicks. B can kick the ball into the Left hand corner, the Right hand corner or into the Middle of the goal. A can Dive to his Right or Dive to his left or Stay where he is. If A correctly guesses where B will kick the ball he will make a save.

The payoff to A is given by the following matrix:

A \ B	R	L	M
DR	2	-1	-2
DL	-1	2	-2
S	-1	-1	1

We shall be considering $m \times n$ generalisations of game 1 and other games like game 2 which can be reduced to this form.

Thus there is given some $m \times n$ payoff matrix $\| a_{ij} \|$.

In a <u>play</u> of the game A chooses $i \in M = \{1, 2, \dots m\}$ and B chooses $j \in N = \{1, 2, \dots n\}$. These choices are made independently without either player knowing what the other has chosen. They then announce their choices and B pays a_{ij} to A.

M, N will be referred to as the sets of <u>tactics</u> for A, B respectively.

A <u>match</u> is an unending sequence of plays. A's objective is to maximise his average winnings from the match and B's objective is to minimise his average losses.

A <u>strategy</u> for the match is some rule for selecting the tactic for the next play.

Let S_A, S_B be sets of strategies for A, B respectively. We shall initially consider the case where $S_A = \{(1), \dots (m)\}$ and $S_B = \{(1), \dots (n)\}$ where (t) is the <u>pure strategy</u> of using tactic t in each play. We shall subsequently be enlarging S_A and S_B and we therefore introduce new notation to allow for this possibility.

Thus for each $u \in S_A$ and $v \in S_B$ let $PAY(u,v)$ denote the average payment of B to A. S_A and S_B will always be such that 'average payment' is meaningful. Thus if $u = (i)$ and $v = (j)$ then $PAY(u, v) = a_{ij}$.

<u>Stable Solutions</u>

$(u_0, v_0) \in S_A \times S_B$ is a <u>stable solution</u> if

(17.1) $\quad PAY(u, v_0) \leq PAY(u_0, v_0) \leq PAY(u_0, v)$.

If (17.1) holds then neither A nor B has any incentive to change strategy if each assumes his opponent is not going to change his.

The subsequent analysis is concerned with finding a stable solution.

Thinking of S_A as the row indices and S_B as the column indices of some matrix we define

$$ROWMIN(u) = \min_{v \in S_B} PAY(u,v) \qquad \text{for } u \in S_A$$

and

$$COLMAX(v) = \max_{u \in S_A} PAY(u,v) \qquad \text{for } v \in S_B$$

Suppose now that A chooses $\hat{u}$. We assume that after some finite time B will be able to deduce this. B will then choose his strategy v to minise $PAY(\hat{u}, v)$. Thus if A chooses u then he can expect his average winnings to be $ROWMIN(u)$.

Similarly if B chooses v he can expect his average losses to be COLMAX(v).

Thus if $P_A = \text{ROWMIN}(u_0) = \max\limits_{u \in S_A} \text{ROWMIN}(u)$ and

$P_B = \text{COLMAX}(v_0) = \min\limits_{v \in S_B} \text{COLMAX}(v)$ then A can by choosing u_0 ensure his winnings average P_A and B by choosing v_0 can ensure his losses average P_B. If $P_A = P_B$ this seems to 'solve' the game but is $P_A = P_B$ always?

<u>Theorem 17.1</u>

(a) $P_A \le P_B$

(b) $S_A \times S_B$ contains a stable solution if and only if $P_A = P_B$

<u>Proof</u>

(a)

(17.2) $P_A = \text{ROWMIN}(u_0) \le \text{PAY}(u_0, v_0) \le \text{COLMAX}(v_0) = P_B$

(b)

Suppose first that $(\hat{u}, \hat{v})$ is stable. Then from (17.1) we have

$$\text{COLMAX}(\hat{v}) = \text{PAY}(\hat{u}, \hat{v}) = \text{ROWMIN}(\hat{u})$$

and hence

$$P_B \le \text{COLMAX}(\hat{v}) = \text{ROWMIN}(\hat{u}) \le P_A$$

which from (a) implies $P_A = P_B$.

Conversely if $P_A = P_B$ then from (17.2) we deduce that $\text{ROWMIN}(u_0) = \text{PAY}(u_0, v_0) = \text{COLMAX}(v_0)$ which implies (17.1).

Q.E.D.

We now consider specifically the case $S_A = \{(1), \cdots (m)\}$ and $S_B = \{(1), \cdots (n)\}$.

For game 1 we have $P_A = P_B = 1 = a_{14}$ and hence A plays (1) and B plays (4) solves the game: A can guarantee to win at least 1 and B can guarantee to lose at most 1 on average.

The matrix of this game is said to have a <u>saddle point</u> (i_o, j_o) which means (i_o), (j_o) satisfies (17.1).

For a game whose matrix does not have saddle point things are more complex. Consider for example game 2. $P_A = -1$ and $P_B = 1$. It follows from theorem 17.1 that no pair of pure strategies solves the game. A knows he can average at least -1 by playing (3) and B knows he need lose no more than 1 on average by playing (3). But note that if A plays (3) then B has an incentive to play (1) or (2). But if B plays (1) A will play (1) and so on.

Mixed Strategies

To break this seeming deadlock we allow the players to choose <u>mixed strategies</u>. A mixed strategy for A is a vector of probabilities $(p_1, \cdots p_m)$ where $p_i \geq 0$ for $i \in M$ and $p_1 + \cdots + p_m = 1$. A then chooses tactic i with probability p_i for $i \in M$ i.e. before each play A carries out a statistical experiment that has an outcome $i \in M$ with probability p_i. A then plays the corresponding tactic. Similarly B's mixed strategies are vectors $(q_1, \cdots q_n)$ satisfying $q_j \geq 0$ for $j \in N$ and $q_1 + \cdots + q_n = 1$.

Pure strategies can be represented as vectors with a single non-zero component equal to 1.

We now 'enlarge' S_A, S_B to

(17.3) $$S_A = \{ \underline{p} \in \mathbb{R}^m : \underline{p} \geq 0 \text{ and } p_1 + \cdots + p_m = 1 \}$$

$$S_B = \{ \underline{q} \in \mathbb{R}^n : \underline{q} \geq 0 \text{ and } q_1 + \cdots + q_n = 1 \}$$

For $\underline{p} \in S_A$, $\underline{q} \in S_B$ it is straightforward to show that

$$PAY(\underline{p}, \underline{q}) = \sum_{i \in M} \sum_{j \in N} a_{ij} p_i q_j$$

We now show using the duality theory of linear programming that $S_A \times S_B$ as defined in (17.3) contains a stable solution.

We shall first show how to compute P_A. Let $c_j(\underline{p}) = \sum_{i \in M} a_{ij} p_i$, then

(17.4) $$P_A = \max_{\underline{p} \in S_A} \left(\min_{\underline{q} \in S_B} \sum_{j=1}^{n} c_j(\underline{p}) q_j \right)$$

Lemma 17.2

(17.5) $$\min_{\underline{q} \in S_B} \left(\sum_{j=1}^{n} \xi_j q_j \right) = \min(\xi_1 \ldots \xi_n)$$

Proof

Let $\xi_t = \min(\xi_1 \cdots \xi_n)$ and let $L =$ LHS of (17.5). Putting $\hat{q}_j = 0$ for $j \neq t$ and $\hat{q}_t = 1$ we have $\hat{\underline{q}} \in Q$ and $\sum_{j=1}^{m} \xi_j \hat{q}_j = \xi_t$.

Thus $L \leq \xi_t$. However for any $\underline{q} \in Q$ we have

$$\sum_{j=1}^{n} \xi_j q_j \leq \sum_{j=1}^{n} \xi_t q_j = \xi_t \sum_{j=1}^{n} q_j = \xi_t .$$

Q.E.D.

It follows from the lemma and (17.4) that

$$P_A = \max_{\underline{p} \in S_A} \left(\min\left(c_1(\underline{p}), \dots c_n(\underline{p}) \right) \right)$$

$$= \max \quad \min\left(c_1(\underline{p}), \dots c_n(\underline{p}) \right)$$

subject to

$$p_1 + \dots + p_m = 1$$

$$p_1, \dots p_m \geq 0$$

(17.6) $$= \max \xi$$

$$\xi \leq \sum_{i=1}^{m} a_{ij} p_i \qquad j = 1, \dots n$$

$$p_1 + \dots + p_m = 1$$

$$p_1, \dots p_m \geq 0$$

where LP (17.6) is derived as in §A of the OR1 notes.

Using similar arguments we can show that

(17.7) $$P_B = \min \eta$$

$$\eta \geq \sum_{j=1}^{n} a_{ij} q_j \qquad i = 1, \dots m$$

$$q_1 + \dots + q_n = 1$$

$$q_1, \dots q_n \geq 0$$

We note next that (17.6) and (17.7) are a pair of dual linear programs. They are both feasible and hence $P_A = P_B$ and stable solutions exist. In fact if $\underline{p}^o$ solves (17.6) and $\underline{q}^o$ solves (17.7) then $(\underline{p}^o, \underline{q}^o)$ is stable as $PAY(\underline{p}^o, \underline{q}^o) = P_A = P_B$.

We describe next a simple transformation which reduces the LP problems by one row and one variable.

Consider first (17.6). We assume that $a_{ij} > 0$. If not we can first add a quantity λ to each a_{ij} so that $a_{ij}+\lambda > 0$. One can see that for any u, v PAY(u,v) is also increased by λ and so this has no effect on the strategies.

We can now assume that $\xi > 0$ in the optimum solution to (17.6). We can the re-write (17.6) as

$$\text{minimise} \quad 1/\xi = \left(\frac{p_1}{\xi}\right) + \cdots + \left(\frac{p_m}{\xi}\right)$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij}\left(\frac{p_i}{\xi}\right) \geq 1 \qquad j = 1, \cdots n$$

(17.8) $$\left(\frac{p_1}{\xi}\right) + \cdots + \left(\frac{p_m}{\xi}\right) = \frac{1}{\xi}$$

$$\left(\frac{p_1}{\xi}\right), \cdots \left(\frac{p_m}{\xi}\right) \geq 0$$

Letting $x_i = p_i/\xi$ for $i = 1, \cdots m$ this becomes

(17.9) $$\text{minimise} \quad x_1 + \cdots + x_m$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} x_i \geq 1 \qquad j = 1 \cdots m$$

$$x_1, \cdots x_m \geq 0$$

where equation (17.8) is redundant.

The optimal strategy $\underline{p}^0$ can be recovered from an optimal solution $\underline{x}^0$ to (17.9) as follows:

let $\xi = 1/(x_1^\circ + \cdots + x_m^\circ)$ and then $p_i^\circ = x_i^\circ \xi$ for $i = 1 \cdots m$.

The equivalent transformation for (17.7) gives

(17.10) maximise $y_1 + \cdots + y_n$

subject to

$$\sum_{j=1}^{n} a_{ij} y_j \leq 1 \qquad i = 1 \cdots m$$

$$y_1, \cdots y_n \geq 0$$

which dual to (17.9)

We now use the above theory to solve game 2.

We first add 3 to each element of the matrix and then solve either (17.9) or (17.10)

Problem (17.10) is

maximise $y_1 + y_2 + y_3$

subject to
$$5y_1 + 2y_2 + y_3 \leq 1$$
$$2y_1 + 5y_2 + y_3 \leq 1$$
$$2y_1 + 2y_2 + 4y_3 \leq 1$$

$$y_1, y_2, y_3 \geq 0$$

The optimal solution can be found to be $y_1 = \frac{1}{8}$ $y_2 = \frac{1}{8}$ $y_3 = \frac{1}{8}$ and $y_1 + y_2 + y_3 = \frac{3}{8}$ implying that $\eta = \frac{8}{3}$ and $q_1 = q_2 = q_3 = \frac{1}{3}$. The solution to (17.9) is $x_1 = \frac{1}{12}$ $x_2 = \frac{1}{12}$ $x_3 = \frac{5}{24}$ implying that $\xi = \frac{8}{3}$ and $p_1 = \frac{1}{7}$ $p_2 = \frac{1}{7}$ and $p_3 = \frac{5}{7}$. Since we added 3 to each element initially we see that the actual value of the game to A is $\frac{8}{3} - 3 = -\frac{1}{3}$.

Dominated Strategies

In the matrix game below

$$\begin{bmatrix} 1 & 3 & -3 & -2 \\ 3 & 2 & 3 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix}$$

we see that strategy (3) is better for A than strategy (1) for any choice of strategy by B and consequently strategy (1) can be ignored by A and the game reduced to

$$\begin{bmatrix} 3 & 2 & 3 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix}$$

We see now that strategy (4) is better than either of strategies (1), (3) for B for either of A's strategies. Thus columns 1, 3 can be deleted to reduce the game to

$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$$

We have used the idea of dominated strategies to reduce the size of the game to be considered. Thus

Strategy (i) <u>dominates</u> strategy (i') for the row player A if

$a_{ij} \geq a_{i'j}$ for all j.

Strategy (j) dominates strategy (j') for the column player B' if

$a_{ij} \leq a_{ij'}$ for all i.

Thus strategies (i'), (j') above can be ignored. Successive applications of these rules can reduce the size of a game significantly.

Random Payoffs

We finally note that the above analysis goes through unchanged if A, B having selected tactics i, j the payoff to A is a random variable whose expected value is a_{ij}

Simple Aspects of games

1 Dominance

If $A(i,j) \geq A(i,j')$ for all i then player B will never use strategy j. It is preferable for her/him to use strategy j' instead. So, column j can be removed from the matrix A.

Similarly, if $A(i,j) \leq A(i',j)$ for all j then player A will never use strategy i. It is preferable for her/him to use strategy i' instead. So, row i can be removed from the matrix A.

Repeated use of this idea can reduce a game substantially.

2 Latin Square Game

Suupose that every row sum is equal to $R > 0$ and every column sum is equal to $C > 0$ where $mR = nC$. Then both players can choose uniformly. Consider the two LP's that solve the game:

$$A \text{ Minimize } \sum_{i=1}^{m} x_i \text{ subject to } \sum_{i=1}^{m} a_{i,j}x_i \geq 1 \text{ for all } j, \sum_{i=1}^{m} x_i = 1. \tag{1}$$

$$B \text{ Maximize } \sum_{j=1}^{n} y_j \text{ subject to } \sum_{j=1}^{n} a_{i,j}j_j \leq 1 \text{ for all } i, \sum_{j=1}^{n} y_j = 1. \tag{2}$$

Putting $x_i = 1/C$ and $y_j = 1/R$ gives two feasible solutions with the same objective value.

3 Non-singular games

Suppose that A is non-singular and that $\mathbf{1}^T A^{-1}\mathbf{1} \neq 0$. Then the value of the game is $V = \frac{1}{\mathbf{1}^T A^{-1}\mathbf{1}}$. Then, $x^T = \frac{\mathbf{1}^T A^{-1}}{V}$ and $y = \frac{A^{-1}\mathbf{1}}{V}$ solve (1), (2) respectively.

4 Symmetric games

A game is symmetric if $A^T = -A$ i.e if A is anti-symmetric. Then the game has value 0. If A and B both use strategy p then because $p^T Ap = 0$ for ant-symmetric A, we see that $PAY(p,p) = 0$. This implies that $0 \geq P_A = P_B \geq 0$.

Appendix 2: Existence of Equilibria in Finite Games

We give a proof of Nash's Theorem based on the celebrated Fixed Point Theorem of L. E. J. Brouwer. Given a set C and a mapping T of C into itself, a point $z \in C$ is said to be a fixed point of T, if $T(z) = z$.

Brouwer's Fixed Point Theorem. *Let C be a nonempty, compact, convex set in a finite dimensional Euclidean space, and let T be a continuous map of C into itself. Then there exists a point $z \in C$ such that $T(z) = z$.*

The proof is not easy. You might look at the paper of K. Kuga (1974), "Brower's fixed point Theorem: An Alternate Proof", *SIAM Journal of Mathematical Analysis,* **5**, 893-897. Or you might also try Parthasarathy and Raghavan (1971), Chapter 1.

Now consider a finite n-person game with the notation of Section III.2.1. The pure strategy sets are denoted by $X_1, \ldots, X_n$, with X_k consisting of $m_k \geq 1$ elements, say $X_k = \{1, \ldots, m_k\}$. The space of mixed strategies of Player k is given by X_k^*,

$$X_k^* = \{p_k = (p_{k,1}, \ldots, p_{k,m_k}) : p_{k,i} \geq 0 \text{ for } i = 1, \ldots, m_k, \text{ and } \textstyle\sum_{i=1}^{m_k} p_{k,i} = 1\}. \quad (1)$$

For a given joint pure strategy selection, $x = (i_1, \ldots, i_n)$ with $i_j \in X_j$ for all j, the payoff, or utility, to Player k is denoted by $u_k((i_1, \ldots, i_n)$ for $k = 1, \ldots, n$. For a given joint mixed strategy selection, $(p_1, \ldots, p_n)$ with $p_j \in X_j^*$ for $j = 1, \ldots, n$, the corresponding expected payoff to Player k is given by $g_k(p_1, \ldots, p_n)$,

$$g_k(p_1, \ldots, p_n) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} p_{1,i_1} \cdots p_{n,i_n} u_k(i_1, \ldots, i_n). \quad (2)$$

Let us use the notation $g_k(p_1, \ldots, p_n|i)$ to denote the expected payoff to Player k if Player k changes strategy from p_k to the pure strategy $i \in X_k$,

$$g_k(p_1, \ldots, p_n|i) = g_k(p_1, \ldots, p_{k-1}, \delta_i, p_{k+1}, \ldots, p_n). \quad (3)$$

where δ_i represents the probability distribution giving probability 1 to the point i. Note that $g_k(p_1, \ldots, p_n)$ can be reconstructed from the $g_k(p_1, \ldots, p_n|i)$ by

$$g_k(p_1, \ldots, p_n) = \sum_{i=1}^{m_k} p_{k,i} g_k(p_1, \ldots, p_n|i) \quad (4)$$

A vector of mixed strategies, $(p_1, \ldots, p_n)$, is a strategic equilibrium if for all $k = 1, \ldots, n$, and all $i \in X_k$,

$$g_k(p_1, \ldots, p_n|i) \leq g_k(p_1, \ldots, p_n). \quad (5)$$

Theorem. *Every finite n-person game in strategic form has at least one strategic equilibrium.*

Proof. For each k, X_k^* is a compact convex subset of m_k dimensional Euclidean space, and so the product, $C = X_1^* \times \cdots \times X_n^*$, is a compact convex subset of a Euclidean space of dimension $\sum_{i=1}^n m_i$. For $z = (p_1, \ldots, p_n) \in C$, define the mapping $T(z)$ of C into C by

$$T(z) = z' = (p_1', \ldots, p_n') \tag{6}$$

where

$$p_{k,i}' = \frac{p_{k,i} + \max(0, g_k(p_1, \ldots, p_n|i) - g_k(p_1, \ldots, p_n))}{1 + \sum_{j=1}^{m_k} \max(0, g_k(p_1, \ldots, p_n|j) - g_k(p_1, \ldots, p_n))}. \tag{7}$$

Note that $p_{k,i} \geq 0$, and the denominator is chosen so that $\sum_{i=1}^{m_k} p_{k,i}' = 1$. Thus $z' \in C$. Moreover the function $f(z)$ is continuous since each $g_k(p_1, \ldots, p_n)$ is continuous. Therefore, by the Brouwer Fixed Point Theorem, there is a point, $z' = (q_1, \ldots, q_n) \in C$ such that $T(z') = z'$. Thus from (7)

$$q_{k,i} = \frac{q_{k,i} + \max(0, g_k(z'|i) - g_k(z'))}{1 + \sum_{j=1}^{m_k} \max(0, g_k(z'|j) - g_k(z'))}. \tag{8}$$

for all $k = 1, \ldots, n$ and $i = 1, \ldots, m_n$. Since from (4) $g_k(z')$ is an average of the numbers $g_k(z'|i)$, we must have $g_k(z'|i) \leq g_k(z')$ for at least one i for which $q_{k,i} > 0$, so that $\max(0, g_k(z'|i) - g_k(z')) = 0$ for that i. But then (8) implies that $\sum_{j=1}^{m_k} \max(0, g_k(z'|j) - g_k(z')) = 0$, so that $g_k(z'|i) \leq g_k(z')$ for all k and i. From (5) this shows that $z' = (q_1, \ldots, q_n)$ is a strategic equilibrium. ∎

Remark. From the definition of $T(z)$, we see that $z = (p_1, \ldots, p_n)$ is a strategic equilibrium if and only if z is a fixed point of T. In other words, the set of strategic equilibria is given by $\{z : T(z) = z\}$. If we could solve the equation $T(z) = z$ we could find the equilibria. Unfortunately, the equation is not easily solved. The method of iteration does not ordinarily work because T is not a contraction map.