Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 4: Due Monday October 28.

1. Give an algorithm to solve the scheduling problem $1|\cdot|\sum_j f(C_j)$ where f is a monotone increasing function.

   Solution: The algorithm here is to schedule the jobs in increasing order of processing times p_i. To see that this is optimal, suppose that job j immediately follows job i in the ordering, but $p_i > p_j$. Suppose that job i starts being processed at time t. Consider the effect on the objective of interchanging i and j. This changes the objective by

$$f(t+p_j) + f(t+p_j+p_i) - f(t+p_i) - f(t+p_i+p_j) \leq 0.$$

2. Give an algorithm to solve the scheduling problem $1|\cdot|\max_j f_j(C_j)$ where f_j is a monotone increasing function for all j.

   Solution: The solution here is to find the job that will go last and then repeat the process for the remaining $n-1$ jobs. Suppose that $P = p_1 + p_2 + \cdots + p_n$ is the total processing time. Then the job that goes last should be the one that minimises $f_j(P)$. We show that this is an optimal decision.

   Suppose that $\Psi(S)$ denotes the optimal solution value for scheduling the jobs in S only. Then by induction, our choice of j produces a solution of cost

$$C = \min_j \max\{\Psi([n] \setminus \{j\}, f_j(P)\}. \tag{1}$$

   On the other hand,

$$\Psi([n]) \geq \min_j\{f_j(P)\} \tag{2}$$

$$\Psi([n]) \geq \Psi([n]) \setminus \{j\}) \tag{3}$$

   Here (2) follows from the fact that some job must go last and complete at time P and (3) follows from the fact that removing a job can only decrease completion times, whatever the order. In addition f is monotone.

So,

$$C \leq \min_j \max\{\Psi([n]), f_j(P)\} = \max\{\Psi([n]), \min_j\{f_j(P)\}\} = \Psi([n]).$$

3. Find the optimal ordering strategy for the following inventory system. If you order an amount Q, it costs AQ^α for some $0 < \alpha < 1$ and the inventory cost is I per unit per period. The demand is λ units per period and stock-outs are allowed. The penalty cost for stock-outs are π per unit per period.

   Solution: The total cost K is given by

$$K = \frac{\lambda A}{Q^{1-\alpha}} + \frac{I(Q-S)^2}{2Q} + \frac{\pi S^2}{2Q}.$$

   We then have, at the minimum,

$$\frac{\partial K}{\partial S} = \frac{I(S-Q)}{Q} + \frac{\pi S}{Q} = 0$$

   which implies that

$$S = \frac{IQ}{I+\pi}.$$

   Then we have,

$$\begin{aligned}
\frac{\partial K}{\partial Q} &= -\frac{\lambda A(1-\alpha)}{Q^{2-\alpha}} + \frac{I(Q-S)}{Q} - \frac{I(Q-S)^2}{2Q^2} - \frac{\pi S^2}{2Q^2} \\
&= -\frac{\lambda A(1-\alpha)}{Q^{2-\alpha}} + \frac{I\pi}{I+\pi} - \frac{I\pi^2}{2(I+\pi)^2} - \frac{\pi I^2}{2(I+\pi)^2} \\
&= -\frac{\lambda A(1-\alpha)}{Q^{2-\alpha}} + \frac{I\pi}{2(I+\pi)} \\
&= 0,
\end{aligned}$$

at the minimum. So, we have

$$
\begin{aligned}
Q &= \left(\frac{2\lambda A(1-\alpha)(I+\pi)}{I\pi}\right)^{1/(2-\alpha)}. \\
S &= \frac{I}{I+\pi}\left(\frac{2\lambda A(1-\alpha)(I+\pi)}{I\pi}\right)^{1/(2-\alpha)}. \\
K &= \lambda A\left(\frac{I\pi}{2\lambda A(1-\alpha)(I+\pi)}\right)^{(1-\alpha)/(2-\alpha)} + \frac{I\pi}{2(I+\pi)}\left(\frac{2\lambda A(1-\alpha)(I+\pi)}{I\pi}\right)^{1/(2-\alpha)} \\
&= \left(\frac{2I\pi}{I+\pi}\right)^{(1-\alpha)/(2-\alpha)} (\lambda A(1-\alpha))^{1/(2-\alpha)}.
\end{aligned}
$$