Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 4: Due Wednesday October 24.

1. Formulate the following as an integer program: There are n students and exams $E_1, E_2, \dots, E_m \subseteq [n]$ need to be scheduled. Here E_i is the set of students that need to take exam i. Exams take place in the morning and in the afternoon. There are s rooms available and each room can hold r students. The rules are (i) A student must not be asked to take more than one exam per day; (ii) Several different exams can be held in the same room at the same time provided there is capacity in the room to hold the students. The problem is to minimise the number of days needed to carry out all of the exams.

   Solution Let

$$x_{i,j,k,m} = \begin{cases} 1 & \text{Exam } i \text{ in room } j \text{ on morning of day } k \\ 0 & \text{Otherwise} \end{cases}$$

$$x_{i,j,k,a} = \begin{cases} 1 & \text{Exam } i \text{ in room } j \text{ on afternoon of day } k \\ 0 & \text{Otherwise} \end{cases}$$

$$y_k = \begin{cases} 1 & \text{There is an exam on day } k \\ 0 & \text{Otherwise} \end{cases}$$

   Then the problem is to

$$\text{Minimise } y_1 + y_2 + \cdots + y_m$$

   subject to

$$x_{i,j,k,m} + x_{i,j,k,a} \leq y_k \qquad \forall i, j, k.$$

$$\sum_{j,k} (x_{i,j,k,m} + x_{i,j,k,a}) = 1 \qquad \forall i.$$

$$\sum_i |E_i| x_{i,j,k,m} \leq r \qquad \forall j, k.$$

$$\sum_i |E_i| x_{i,j,k,a} \leq r \qquad \forall j, k.$$

$$\sum_j (x_{i,j,k,m} + x_{i',j,k,m} + x_{i,j,k,a} + x_{i',j,k,a}) \leq 1 \qquad \forall k, \forall i, i' : E_i \cap E_{i'} \neq \emptyset.$$

2. Solve the following problem by a cutting plane algorithm:

$$\begin{array}{llllll} \text{minimise} & 4x_1 & + & 5x_2 & +3x_3 & \\ \text{subject to} & & & & & \\ & 2x_1 & + & x_2 & -x_3 & \geq 2 \\ & x_1 & + & 4x_2 & +x_3 & \geq 13 \end{array}$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution

Initial tableau

x_1	x_2	x_3	x_4	x_5	R.H.S	
-4	-5	-3	0	0	0	z
-2	-1	1	1	0	-2	x_4
-1	-4	-1	0	1	-13	$x_5 \leftarrow$
	$\uparrow$					

x_1	x_2	x_3	x_4	x_5	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	$\frac{13}{4}$	x_2

Primal feasible, but the solution is not integral.

We add a cut which eliminates the current optimal solution.

$$\frac{1}{4}x_1 + \frac{1}{4}x_3 + \frac{3}{4}x_5 - y_1 = \frac{1}{4}$$

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	0	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	0	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	0	$\frac{13}{4}$	x_2
$\frac{-11}{4}$	0	$\frac{-1}{4}$	0	$\frac{-3}{4}$	+1	$\frac{-1}{4}$	$y_1 \leftarrow$
				$\uparrow$			

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	0	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	$\frac{-1}{3}$	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{-1}{3}$	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	$\frac{-4}{3}$	$\frac{1}{3}$	x_5

The solution is primal feasible and so optimal but still not integer. We add a cut which eliminates the current optimal solution.

$$\frac{-1}{3}x_1 - \frac{1}{3}x_3 + y_2 = \frac{1}{3}$$

We obtain tableau

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	0	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	0	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	x_5
$\frac{-1}{3}$	0	$\frac{-1}{3}$	0	0	1	$\frac{1}{3}$	$y_2 \leftarrow$
		$\uparrow$					

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
-1	0	0	0	0	-4	18	z
-3	0	0	1	0	4	0	x_4
0	1	0	0	0	1	3	x_2
0	0	0	0	1	1	0	x_5
1	0	1	0	0	-3	1	x_3

Which is optimal integral.

3. Solve the following problem by a branch and bound algorithm:

$$\begin{array}{llllll}
\text{Maximise} & 4x_1 & -2x_2 & +7x_3 & -x_4 & \\
\text{subject to} & & & & & \\
 & x_1 & & +5x_3 & & \leq 10 \\
 & x_1 & +x_2 & -x_3 & & \leq 1 \\
 & 6x_1 & -5x_2 & & & \leq 0 \\
 & -x_1 & & +2x_3 & -2x_4 & \leq 3
\end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

$$x_1, x_2, x_3 \text{ integer.}$$

Solution

1. LP relaxation:

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 0\right) \qquad Value = 14\frac{1}{4}.$$

Sub-problem 1: add constraint $x_1 \leq 1$.

$$(x_1, x_2, x_3, x_4) = \left(1, \frac{6}{5}, \frac{9}{5}, 0\right) \qquad Value = 14\frac{1}{5}.$$

Sub-problem 2: add constraint $x_1 \geq 2$.
No solutions.

Subproblem 1.1: add constraint $x_2 \leq 1$.

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 1, \frac{11}{6}, 0\right) \qquad Value = 14\frac{1}{6}.$$

Subproblem 1.2: add constraint $x_2 \geq 2$.

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 2, \frac{11}{6}, 0\right) \qquad Value = 12\frac{1}{6}.$$

Sub-problem 1.1.1: add constraint $x_1 \leq 0$.

$$(x_1, x_2, x_3, x_4) = \left(0, 0, 2, \frac{1}{2}\right) \qquad Value = 13\frac{1}{2}.$$

This solution is feasible.

Subproblem 1.1.2: add constraint $x_1 \geq 1$. No solutions.

Sub-problem 1.2 is *fathomed* i.e. there is no solution to this problem which is better than our current *incumbent*.

Optimal solution: $(x_1, x_2, x_3, x_4) = \left(0, 0, 2, \frac{1}{2}\right) \qquad Value = 13\frac{1}{2}$.