10\21\13

Two person Zero-Sum Games

$$A \backslash B \quad \begin{bmatrix} 2 & 4 & 2 & 1 \\ -2 & 5 & 1 & -1 \\ 1 & -5 & 3 & 0 \\ 6 & 2 & -3 & -2 \end{bmatrix}$$

A chooses a row i.

B chooses a column j

Neither know what the other choice is while choosing their own.

B pays A, a_{ij}.

If $a_{ij} < 0$ then A pays $-a_{ij}$ to B.

Question: how should games like this be played?

Soccer: Penalty Kick

B Goal keeper

A Kicker

$$
\begin{array}{r c}
 & \begin{array}{ccc} \text{Stay} & \text{Dive Left} & \text{Dive Right} \end{array} \\
\begin{array}{r} \text{Kick Straight} \\ \text{Left} \\ \text{Right} \end{array} & \begin{bmatrix} -1 & +1 & +1 \\ +1 & -1 & +1 \\ +1 & +1 & -1 \end{bmatrix}
\end{array}
$$

Rock_Paper_Scissors

$$\begin{array}{c|c} & \begin{matrix} R & P & S \end{matrix} \\ \begin{matrix} R \\ P \\ S \end{matrix} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

A & B play an unending sequence of games.

A wants to maximise average winnings per game

B wants to minimise average winning per game

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 12 & 8 & 11 \\ 21 & 5 & 26 \\ 1 & 7 & -1 \end{bmatrix} \end{array}$$

8 is a saddle point

This maximises A's guaranteed winnings and minimises B guaranteed losses.

S_B = set of columns

S_A = set of rows

$$\begin{array}{cc} & v \\ u & \left[\begin{array}{c} \\ \mathrm{PAY}(u,v) \\ \\ \end{array}\right] \end{array}$$

$S_A = \{A\text{'s strategies}\}$

$S_B = \{B\text{'s strategies}\}$

$\mathrm{PAY}(u,v)$ = average gain for A if A plays u and B plays v.

Stable Solution

(u_0, v_0) is stable if

$$\text{PAY}(u, v_0) \leq \text{PAY}(u_0, v_0) \leq \text{PAY}(u_0, v)$$

$\forall u$ $\qquad\qquad\qquad\qquad\qquad\qquad \forall v$

$$\mathrm{ROWMIN}(u) = \min_{v \in S_B} \mathrm{PAY}(u,v)$$

$$\mathrm{COLMAX}(v) = \max_{u \in S_A} \mathrm{PAY}(u,v)$$

P_A: A's guaranteed winnings: $= \max_{u \in S_A} \mathrm{ROWMIN}(u)$

P_B: B's minimum guaranteed losses $= \min_{v \in S_B} \mathrm{COLMAX}(v)$

__Thm__

(i) $\rho_A \leq \rho_B$

(ii) $\rho_A = \rho_B$ iff $S_A \times S_B$ contains a stable solution.

__Proof__

(i) $\mathrm{ROWMIN}(u) \leq \mathrm{PAY}(u, v) \leq \mathrm{COLMAX}(v)$