2 person, zero sum games

\[ \begin{array}{c}
i \\
i \end{array} \begin{bmatrix}
\mathbf{A}_{ij} \\
\mathbf{A}_{ij}
\end{bmatrix} \]

2 players A, B

A chooses i, B chooses j (without each other's knowledge)

B pays A, a_{ij}
Rock, Paper, Scissors

\[
\begin{bmatrix}
R & P & S \\
R & 0 & -1 & 1 \\
P & 1 & 0 & -1 \\
S & -1 & 1 & 0
\end{bmatrix}
\]

Penalty Kicks in Soccer

\[
\begin{bmatrix}
L & C & R \\
\text{Player} & -2 & 1 & 1 \\
\text{Keeper} & 2 & -1 & 2 \\
\text{Shooter} & 1 & 1 & -2
\end{bmatrix}
\]
Games arise anywhere there is conflict: economics, war, animal behavior.

Assume an unending sequence of plays of the game

Strategy is some rule for playing the game.
\[ S_A = \{ A \text{'s strategies} \} \]
\[ S_B = \{ B \text{'s strategies} \} \]

Pure strategy for A: (i) 
Always play i

We expand set of strategies beyond just pure strategies.
\( u \in S_A \)
\( v \in S_B \)

\[ \text{average} \]
\[ \text{PAY}(u,v) = \text{payoff to A} \]
\[ P((i_j),(i_j)) = a_{i_j} \]

Two ways of viewing:

(i) Stable Solution:

(\( u_0, v_0 \)) in stable if

\[ \text{PAY}(u_j, v_0) \leq \text{PAY}(u_0, v_0) \leq \text{PAY}(u_0, v) \]

?? Does \( \exists \) a stable solution ??
If $S_A = \{1, 2, \ldots, n\}$
and $S_B = \{1, 2, \ldots, n\}$
then there may not be a stable solution.

\[
\begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{bmatrix}
\]
(11) Both players are cautious:
If A plays \( u \), can definitely get
\[ \text{rowmin}(u) = \min_v \text{pay}(u, v) \]
A chooses \( \max_u \text{rowmin}(u) \leftarrow P_A \)
B chooses \( \min_v \text{colmax}(v) \leftarrow P_B \)

Both notions lead to same strategy
here.

\[ P_A \leq P_B \]
We show next

\[ P_A = P_B \iff \exists \text{ stable solution.} \]

\[
\begin{bmatrix}
2 & 3 & 4 \\
1 & 5 & 8 \\
1 & -2 & -7
\end{bmatrix}
\]

Mixed strategy:

A chooses vector \( p = (p_1, p_2, \ldots, p_m) \) s.t. \( p_i \geq 0, p_1 + \ldots + p_m = 1 \)

B chooses a vector \( q = (q_1, q_2, \ldots, q_n) \) s.t. \( q_1 + \ldots + q_n = 1 \)

Next play for A in \( i \) with prob. \( p_i \), etc.
\[ \text{PAY}(\rho, \varphi) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \rho \cdot \varphi_i \]

\[ S_A = \xi \rho \xi \]

\[ S_B = \xi \varphi \xi \]