HIGHER DIMENSIONAL DISTORTION OF RANDOM COMPLEXES

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Abstract. Using the random complexes of Linial and Meshulam [19], we exhibit a large family of simplicial complexes for which, whenever affinely embedded into Euclidean space, the filling areas of simplicial cycles is greatly distorted. This phenomenon can be regarded as a higher order analogue of the metric distortion of embeddings of random graphs.

1. Introduction

One of the natural questions to ask when we come across a new geometric object is "How does it compare to Euclidean space?" We examine objects from this viewpoint not only because we all live in Euclidean space, but also because being a subset of Euclidean space is one of the most stringent geometric conditions cf. [8], [21], [22]. The study of how well discrete and continuous objects "fit" into Euclidean space, Banach spaces, or geometric space forms is extensive (the literature is vast but the author might suggest [18], or [20], as a place to start). Finding good embeddings (or obstructions to them) for discrete structures (graphs, groups, finite metric spaces, etc.) into L^2 or L^1 is an industry in its own right, particularly because of the direct applications to theoretical computer science.

1.1. Metric distortion.

We begin by giving a seminal example of the quantitative study of obstructions to "nice" embeddings:

Theorem (J. Bourgain, [5])**.**

Denote by $G_{n,p}$ the Erdős–Rényi random graph on n vertices, i.e. the probability space of graphs on n vertices with

$$\mathbb{P}[G \in G_{n,p}] = p^E(1-p)^{\binom{n}{2}-E}$$

where E is the number of edges in G.

If $p = \frac{K \log n}{n}$, then with probability tending to 1 as $n \to \infty$, any embedding $\phi : G \to \mathcal{H}$ of $G \in G_{n,p}$ into a Hilbert space must have

$$\max_{x,y \in X} \frac{\|\phi(x) - \phi(y)\|}{d(x,y)} \cdot \max_{x,y \in X} \frac{d(x,y)}{\|\phi(x) - \phi(y)\|} \geq C \frac{\log n}{\log \log n}.$$

The left side of the inequality is referred to as the *metric distortion* of ϕ (we will denote it by $\delta_0(\phi)$). Metric distortion of embeddings is one of many avenues in which random constructions, such as Erdős–Rényi random graphs, have been of great value to geometry. It is somewhat paradoxical that random (generic) objects can be good examples of extremal geometries.

However, we have no intention of adding to this very lively discussion of the theory of metric embeddings. Instead we will be most interested in *higher order* phenomena, namely the properties of embeddings of random simplicial complexes into Euclidean space. First, then, we must decide what we mean by "higher dimensional metric distortion."

1.2. **Filling distortion.** In this paper, we will be seeking obstructions to the existence of embeddings of geometric objects which preserve some higher geometric structure. With this in mind we propose the following definition.

Definition 1.1.

Let X be a simplicial complex and $\phi : X \to \mathcal{H}$ a Lipschitz map from X to a Hilbert space. The *filling distortion* of ϕ is given by:

$$\delta_1(\phi) = \sup_{z \in Z_1 X} \frac{\mathrm{Fill}_{\mathcal{H}}\, \phi_* z}{\mathrm{Fill}_X\, z} \cdot \sup_{z \in Z_1 X} \frac{\mathrm{Fill}_X\, z}{\mathrm{Fill}_{\mathcal{H}}\, \phi_* z}$$

where $\mathrm{Fill}(z)$ is the size of the smallest chain, y, with $\partial y = z$.

Namely, the filling distortion measures how badly the filling areas of 1-cycles is distorted in $\mathcal{H}$ (in turn, this can be thought of as the *metric distortion* of the induced map $\phi_* : Z_1 X \to Z_1 \mathcal{H}$, each endowed with the *flat metric* [26]). We will discuss this definition in more detail in the final section.

With a definition in place, we can then look for a candidate complex to achieve a high level of distortion. Our main theorem addresses this:

Theorem 1.2.

For every large n and for $\epsilon > 0$, there exists a 2-dimensional simplicial complex on n vertices, with complete 1-skeleton ($\binom{n}{2}$ edges), with the property that any affine map $\phi : X \to \mathcal{H}$ into a Hilbert space must have

$$\delta_1(\phi) \geq C n^{\frac{1-2\epsilon}{4}}.$$

We would like to draw the reader's attention to the fact that, although we have chosen the definition to complement the notion of metric distortion, it turns out that these quantities reflect very different phenomena. Our main theorem gives a lower bound in terms of a *power* (rather than the logarithm) of n.

In the course of proving our main theorem, we will prove an intermediate proposition:

Proposition 1.3.

Let X be a 2-dimensional simplicial complex on n vertices, a complete 1-skeleton and the smallest eigenvalue of the Laplacian acting on 1-forms given by $\lambda^1(X)$. Then for any affine map $\phi : X \to \mathcal{H}$, suitably scaled so that

$$\mathrm{Fill}_{\mathcal{H}}(\phi\tau) \geq \mathrm{Fill}_X(\tau)$$

for every triangle, τ. Then,

$$\sum_{f \in X^{(2)}} \big(\mathrm{Area}(\phi f)\big)^2 \geq \frac{\lambda^1(X)}{3(n-2)} \sum_{\tau} \big(\mathrm{Fill}_X \tau\big)^2.$$

where the first sum runs over 2-dimensional faces in X and the second sum runs over all triangles in X.

With this inequality in mind, we will attempt to maximize the quantity,

$$\frac{\lambda^1(X)}{|X^{(2)}|} \sum_{\tau} \big(\mathrm{Fill}_X \tau\big)^2$$

Just as Bourgain did with random graphs in [5], we rely on random complexes:

Definition (Linial and Meshulam, [19])**.**

Let $Y_{n,p}$ denote the *Linial–Meshulam random complex*, the probability space of 2-dimensional simplicial complexes on n vertices, with complete 1-skeleton (i.e. $\Delta_n^{(1)} \subset Y \subset \Delta_n^{(2)}$), such that

$$\mathbb{P}[Y \in Y_{n,p}] = p^F (1-p)^{\binom{n}{3} - F} \qquad \text{where } F \text{ is the number of faces in } Y.$$

The random complexes of Linial and Meshulam end up giving the estimate in theorem 1.2 with 99% probability.

1.3. **Other results on maps to Euclidean space.** There have already been some recent results on maps from random simplicial complexes to Euclidean space of a topological, rather than explicitly geometric, nature. We have two in particular in mind. The first is Gromov's point selection theorem for random complexes:

Theorem (Gromov, [12] (combined with the main observation in [6]))**.**

There exists a constant, c, such that if $p > \frac{K \log n}{n}$ then with probability tending to 1 as n tends to infinity, $Y \in Y_{n,p}$ has the property that for any continuous map $\phi : Y \to \mathbb{R}^2$, there exists a point $p \in \mathbb{R}^2$ which lies in the image of at least $c|Y^{(2)}|$ of the 2-dimensional faces of Y.

This theorem simply says that a Linial–Meshulam random complex, once it has enough faces, will tend to pile up, or "overlap" ([11]) when mapped into $\mathbb{R}^2$.

Another recent result addresses the question of topological embeddability:

Theorem (Wagner, [25]).

If $p > \frac{K}{n}$, then with probability tending to 1 as n tends to infinity, $Y \in Y_{n,p}$ does not topologically embed in $\mathbb{R}^4$

This is the higher dimensional analogue of the fact that Erdős–Rényi random graphs are overwhelmingly non-planar (for a large enough p).

The reader should be informed that we did not state either of these theorems in nearly their highest generality, but instead offered them in this form to lend them better to the theme of this article.

1.4. **Overview.** In the next section we will develop our notation and be more explicit with our definitions. Once our notation is in place we will prove proposition 1.3 in section 3. We will follow up in section 4 by estimating the desired spectral and isoperimetric quantities of Linial–Meshulam random complexes, thereby obtaining theorem 1.2. We reserve the final section for a series of questions and remarks.

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2. Notation and concepts

Throughout, we will concern ourselves with a simplicial complex, X, whose k-dimensional faces are denoted $X^{(k)}$. This induces two chain complexes of interest:

$$0 \to C^{-1}_{\mathbb{R}}X \xrightarrow{d} C^{0}_{\mathbb{R}}X \xrightarrow{d} C^{1}_{\mathbb{R}}X \xrightarrow{d} C^{2}_{\mathbb{R}}X \to 0$$

where $C^k_{\mathbb{R}}X = \{X^{(k)} \to \mathbb{R}\}$ and for $\beta \in C^{k-1}_{\mathbb{R}}X$, $d\beta(y) = \sum_{x \in \partial y} \beta(x)$.

For real cochains, the norm is understood to be $\|\beta\| = \sqrt{\sum_{x \in X^{(k)}} |\beta(x)|^2}$.

For a simplicial complex, X, we define the (up-down) k-th spectral gap to be

$$\lambda^k(X) = \inf_{\beta \in C^k_{\mathbb{R}}X} \frac{\|d\beta\|^2}{\inf_{\alpha \in C^{k-1}_{\mathbb{R}}X} \|\beta + d\alpha\|^2} = \inf_{\beta,\, \partial\beta = 0} \frac{\|d\beta\|^2}{\|\beta\|^2}$$

(see [15], [10], [23] for more on the combinatorial Hodge decomposition, and see [6] for more on coboundary expansion).

We will also consider the chain complex with $\mathbb{Z}_2$-coefficients:

$$0 \leftarrow C_{-1}X \xleftarrow{\partial} C_0X \xleftarrow{\partial} C_1X \xleftarrow{\partial} C_2X \xleftarrow{\partial} 0.$$

This chain complex will be endowed with the L^1 (or *Hamming*) norm, $\|y\| = |\mathrm{supp} y|$.

We will denote the space of k-cycles by $Z_kX = \ker \partial_k \subset C_kX$. The space of cycles has a natural metric, the *flat metric* [26]:

$$\|z\|_\flat = \mathrm{Fill}_X(z) = \inf\{\|y\| : \partial y = z\}.$$

3. Dilation estimates for embeddings

The section will be devoted to proving our main proposition:

Proposition. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a 2-dimensional simplicial complex with complete 1-skeleton. Let $\phi : X \to \mathcal{H}$ be an affine embedding of X into an infinite dimensional Hilbert space, suitably scaled so that*

$$\mathrm{Fill}_{\mathcal{H}}(\psi\tau) \geq \mathrm{Fill}_X(\tau)$$

for every triangle, τ, then,

$$\sum_{f\in X^{(2)}} \big(\mathrm{Area}(\phi f)\big)^2 \geq \frac{\lambda^1(X)}{3(n-2)} \sum_{\tau} \big(\mathrm{Fill}_X \tau\big)^2.$$

where the first sum runs over 2-dimensional faces in X and the second sum runs over all triangles in X.

Proof. We can assume that the image of the 0-skeleton, $X^{(0)}$ forms a linearly independent set in $\mathcal{H}$ (since a small perturbation of the vertices does not change the areas of triangles very much).

Choose orthonormal coordinates, $x_1, \dots, x_n$ for $\mathrm{span}X^{(0)} \cong \mathbb{R}^n$. We will let ϕ induce a function $\psi : X^{(1)} \to \mathbb{R}^{\binom{n}{2}}$ defined by

$$\psi_{(i<j)}(e) = \frac{1}{2}\int_{\phi(e)} x_i dx_j - x_j dx_i + \sum_{m=0}^{n} \int_{\phi(e)} y_m^{(i<j)} dx_m \quad \text{for each } e \in X^{(1)}$$

(with the $y_m^{(i<j)}$ as fixed constants to be chosen later).

Now we have $d\psi : X^{(2)} \to \mathbb{R}^{\binom{n}{2}}$, and by the Stokes theorem,

$$(d\psi(f))_{(i<j)} = \int_{\phi f} dx_i \wedge dx_j.$$

Now it is easily seen that $|d\psi(f)|^2 = \big(\mathrm{Area}(\phi f)\big)^2$. This is because the area form of ϕf can we written as

$$\omega_{\phi f} = \sum_{i<j} a_{(i<j)} dx_i \wedge dx_j \;\text{ where }\; \sum_{i<j} a_{(i<j)}^2 = 1 \;\text{ and }\; \int_{\phi f} dx_i \wedge dx_j = a_{(i<j)}\mathrm{Area}(\phi f).$$

Thus,

$$\|d\psi\|^2 = \sum_{f\in X^{(2)}} \big(\mathrm{Area}(\phi f)\big)^2.$$

We will need to prove a small claim:

Claim. Consider the function, $\xi : X^{(1)} \to \mathbb{R}^{\binom{n}{2}}$ given by

$$(\xi(e))_{(i<j)} = \frac{1}{2}\int_{\phi(e)} x_i dx_j - y_j dx_i.$$

Then for any $\alpha : X^{(0)} \to \mathbb{R}^{\binom{n}{2}}$, we can choose $y_m^{(i<j)}$ so that $\psi = \xi + d\alpha$.

Proof. If $e = [v, w] \in X^{(1)}$,

$$\sum_m \int_{\phi(e)} y_m^{(i<j)} dx_m = \langle y^{(i<j)}, \phi(v)\rangle - \langle y^{(i<j)}, \phi(w)\rangle$$

Since $\phi(X^{(0)})$ is linearly independent, for every function $f : X^{(0)} \to \mathbb{R}$, there exists a corresponding $y \in \mathbb{R}^n$ such that

$$f(x) \equiv \langle y, \phi(x)\rangle \quad \text{ for every } x \in X^{(0)}.$$

Therefore, for every function, $f : X^{(0)} \to \mathbb{R}^{\binom{n}{2}}$ we can choose $\binom{n}{2}$ such $y \in \mathbb{R}^n$ (denoted $y^{(i<j)}$) such that

$$(f_{(1<2)}(x), \ldots, f_{(n-1<n)}(x)) \equiv (\langle y^{(1<2)}, \phi(x)\rangle, \ldots, \langle y^{(n-1<n)}, \phi(x)\rangle \text{ for every } x \in X^{(0)}.$$

□

Since we can choose $(y_1, \ldots, y_n)$ so that $\partial\psi = 0$, we have the inequality:

$$\sum_{f\in X^{(2)}} \big(\mathrm{Area}(\phi f)\big)^2 = \|d\psi\|^2 \geq \lambda^1(X)\|\psi\|^2.$$

Now we have only to prove that

$$\|\psi\|^2 \geq \frac{1}{3(n-2)} \sum_\tau \big(\mathrm{Fill}_X(\tau)\big)^2.$$

We observe that for a triangle τ formed by the edges e_1, e_2, and e_3, we have (by the Stokes theorem again),

$$\Big[\psi(e_1) + \psi(e_2) + \psi(e_3)\Big]_{(i<j)} = \int_{\phi\tau} dx_i \wedge dx_j$$

So that,

$$\sum_{(i<j)} \Big[\psi(e_1)+\psi(e_2)+\psi(e_3)\Big]^2_{(i<j)} = \mathrm{Fill}_{\mathcal{H}}(\phi\tau) \geq \mathrm{Fill}_X(\tau)$$

and by Cauchy-Schwartz:

$$|\psi(e_1)|^2+|\psi(e_2)|^2+|\psi(e_3)|^2 \geq \frac{1}{3}\big(|\psi(e_1)|+|\psi(e_2)|+|\psi(e_3)|\big)^2.$$

Summing over all triangles, each edge is contained in $n-2$ triangles, we have

$$(n-2)\|\psi\|^2 \geq \frac{1}{3}\sum_{\tau}\big(\mathrm{Fill}_X(\tau)\big)^2.$$

□

In light of this proposition, it should be clear to the reader that we seek to find 2-dimensional complexes which maximize the quantity,

$$\frac{\lambda^1(X)}{|X^{(2)}|}\sum_{\tau}\big(\mathrm{Fill}_X(\tau)\big)^2.$$

As we increase the number of 2-faces, $\lambda^1(X)$ will go up, but $\frac{\sum(\mathrm{Fill}_X(\tau))^2}{|X^{(2)}|}$ will go down.

It is not clear to the author how to build exact optimizers for this quantity, so in the next section we will resort to using random complexes a la Linial and Meshulam [19].

4. Filling estimates for random complexes

Since we have given ourselves the liberty to take estimates up to a constant, we will exhibit a somewhat cavalier indifference to preserving sharp quantities. We have decided to err on the side of keeping things simple and believe the reader will forgive us for this minor transgression.

We will rely on the geometry of random complexes. We recall the definition:

Definition (Linial and Meshulam, [19])**.**

Let $Y_{n,p}$ denote the *Linial–Meshulam random complex*, the probability space of 2-dimensional simplicial complexes on n vertices, with complete 1-skeleton (i.e. $\Delta_n^{(1)} \subset Y \subset \Delta_n^{(2)}$, where Δ_n denotes the complete simplicial complex, the $(n-1)$-dimensional simplex), such that

$$\mathbb{P}[Y \in Y_{n,p}] = p^F(1-p)^{\binom{n}{3}-F} \qquad \text{where } F \text{ is the number of faces in } Y.$$

Proposition 4.1. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a p-random complex. There is a constant, C, so that if $p \geq \frac{C\log n}{n}$, then, with probability tending to 1 as $n \to \infty$,*

$$\lambda^1(X) \geq \frac{1}{3}pn.$$

Proof. The proof of this proposition is a simple consequence of theorem 2 in [13]:

Theorem (Gundert and Wagner, [13])**.** *Let $\hat{\lambda}^1(X)$ denote the spectral gap of normalized Laplacian on 1-forms. For all $c > 0$, there exists a constant K such that if $p \geq \frac{K \log n}{n}$ and $X = X_{n,p}$ is a random 2-complex (with complete 1-skeleton) then*

$$\hat{\lambda}^1(X; \mathbb{R}) \geq 1 - \frac{K}{\sqrt{pn}}$$

with probability greater than $1 - n^{-c}$.

In order to prove proposition 4.1, we simply need to prove the following claim:

Claim. With probability tending to 1 as n tends to infinity, the degree of each edge is greater than $\frac{p(n-2)}{2}$.

Proof. Our argument is a standard one.

The expected degree of each edge is $p(n-2)$. By a form of Chernoff's inequality [7], each edge, e, has

$$\mathbb{P}[\deg(e) < (1-\epsilon)p(n-2)] \leq e^{-\frac{\epsilon^2 p(n-2)}{2}}.$$

Taking $\epsilon = \frac{1}{2}$, and taking a union bound:

$$\sum_e \mathbb{P}\Big[\deg(e) < \frac{p(n-2)}{2}\Big] \leq e^{\frac{p(n-2)}{8}} \leq \binom{n}{2} \cdot n^{-\frac{K}{8}} \to 0 \qquad \text{by taking } K > 16.$$

□

Now applying this to the theorem of Gundert and Wagner (since the normalized Laplacian is obtained simply by dividing the differential of an edge by its degree), we have

$$\lambda(X) \geq \frac{p(n-2)}{2} - K\sqrt{pn} \geq \frac{pn}{3} \qquad \text{for large } n.$$

□

Now we are left to find an lower bound on $\sum_\tau \big(\mathrm{Fill}_X(\tau)\big)^2$.

Proposition 4.2. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a p-random complex with $p = n^{\epsilon - 1}$, then, with probability tending to 1 as $n \to \infty$,*

$$\sum_\tau \big(\mathrm{Fill}_X(\tau)\big)^2 \geq Cn^{4-2\epsilon}$$

Proof. First, shall examine a single triangle, τ, and bound the probability that $\mathrm{Fill}(\tau) < n^\alpha$. To achieve this, we appeal to an estimate made in [3] (later revised to [2]), but attributed as an observation of Eran Nevo, that a k-cycle in $z \in Z_k\Delta_n$ which does not contain any smaller cycles as a subset and which is supported on $f_0(z)$ vertices and $f_d(z)$ faces of dimension d must have

$$f_0 \leq \frac{f_d + (d+2)(d-1)}{d}.$$

Now, a minimal filling (i.e. does not contain a smaller filling as a subset) of τ can be obtained from a minimal cycle, z, which contains the face which τ bounds. Thus, the number of fillings of τ of size m in Δ_n can be bounded by,

$$\begin{aligned} \binom{n}{f_0-d-1}\binom{\binom{f_0}{d+1}}{m} &\leq n^{f_0-d-1}\left(\frac{ef_0^{d+1}}{m}\right)^m \\ &\leq n^{\frac{m+(d+2)(d-1)-d^2-d}{d}}(Cm^d)^m \\ &= n^{-\frac{2}{d}}(Cn^{\frac{1}{d}}m^d)^m \end{aligned}$$

Therefore, setting $d=2$, we have,

$$\mathbb{P}[\exists y,\, \partial y = \tau,\, \|y\| < n^\alpha] \leq n^{-1}\sum_{m\geq 3}^{n^\alpha}(Cpn^{\frac{1}{2}}m^2)^m \leq n^{-1}\sum_{m\geq 3}(en^{2\alpha+\epsilon-\frac{1}{2}})^m$$

So that,

$$\mathbb{P}[\exists y,\, \partial y = \tau,\, \|y\| < n^\alpha] \;\leq\; Cn^{3(2\alpha+\epsilon-\frac{1}{2})-1}\frac{n^{(n^\alpha-2)(2\alpha+\epsilon-\frac{1}{2})}-1}{n^{2\alpha+\epsilon-\frac{1}{2}}-1}$$

Therefore, if we set $2\alpha+\epsilon-\frac{1}{2} < 0$, then we have

$$\mathcal{P}[\mathrm{Fill}_X(\tau) < n^\alpha] \to 0$$

and

$$\mathbb{E}[\mathrm{Fill}_X\tau] \geq cn^{\frac{1-2\epsilon}{4}} \qquad \text{for large enough } n.$$

Now we will bound the quantity

$$\mathbb{E}_Y\left[\sum_\tau \min\{\mathrm{Fill}_X(\tau), n^{\frac{1-2\epsilon}{4}}\}\right].$$

On the one hand,

$$\mathbb{E}_Y\left[\sum_\tau \min\{\mathrm{Fill}_X(\tau), n^{\frac{1-2\epsilon}{4}}\}\right] \geq \binom{n}{3}n^{\frac{1-2\epsilon}{4}}\mathbb{P}[\mathrm{Fill}_X(\tau) \geq n^{\frac{1-2\epsilon}{4}}] \geq \frac{99}{100}\binom{n}{3}n^{\frac{1-2\epsilon}{4}}.$$

On the other hand, if we let

$$H = \left\{Y \in Y_{n,p} :\ \text{at least } \frac{1}{100}\binom{n}{3} \text{ triangles have } \mathrm{Fill}_X\tau \geq \frac{n^{\frac{1-2\epsilon}{4}}}{99}\right\}$$

(Notice that H implies the proposition),

then,

$$\binom{n}{3}n^{\frac{1-2\epsilon}{4}}\mathbb{P}[H] + \left[\frac{99}{100}\binom{n}{3}\frac{n^{\frac{1-2\epsilon}{4}}}{99} + \frac{1}{100}\binom{n}{3}n^{\frac{1-2\epsilon}{4}}\right]\mathbb{P}[H^C] \geq \mathbb{E}_Y\left[\sum_\tau \min\{\mathrm{Fill}_X(\tau), n^{\frac{1-2\epsilon}{4}}\}\right]$$

So therefore, we have:

$$\mathbb{P}[H] + \frac{1}{50}(1 - \mathbb{P}[H]) \geq \frac{99}{100} \Rightarrow \mathbb{P}[H] \geq \frac{97}{98}.$$

$\square$

Corollary 4.3. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a p-random complex with $p = n^{\epsilon-1}$. Then with probability tending to 1 as $n \to \infty$, every affine embedding $\phi : X \to \mathcal{H}$ of X into an infinite dimensional Hilbert space, $\mathcal{H}$ must have,*

$$\max_{z \in Z_1 X} \frac{\mathrm{Fill}_{\mathcal{H}}(\phi_* z)}{\mathrm{Fill}_X(z)} \cdot \max_z \frac{\mathrm{Fill}_X(z)}{\mathrm{Fill}_{\mathcal{H}}(\phi_* z)} \geq C n^{\frac{1-2\epsilon}{4}}$$

Proof. Taking an affine map, $\phi : X \to \mathcal{H}$ and scaling it so that $\mathrm{Fill}_{\mathcal{H}}(\phi_* \tau) \geq \mathrm{Fill}_X(\tau)$ (again, the filling distortion is continuous with respect to small perturbations, so we may always perturb ϕ slightly and then scale it as prescribed). Then there is a 2-face of X such that:

$$\big(\mathrm{Area}(\phi f)\big)^2 \geq \frac{\lambda^1(X)}{3(n-2)|X^{(2)}|} \sum_\tau \big(\mathrm{Fill}_X \tau\big)^2 \geq c\frac{1}{n^3} \sum_\tau \big(\mathrm{Fill}_X \tau\big)^2 \geq c n^{\frac{1-2\epsilon}{2}}$$

Therefore,

$$\delta_1(\phi) \geq c n^{\frac{1-2\epsilon}{4}}.$$

$\square$

5. Remarks and questions

In this section, we give a few remarks and a few open questions.

5.1. **Remarks on defintion 1.1.**

It may be that our definition 1.1 needs further explanation and motivation.

If X is a geodesic space (such as the geometric realization of a metric graph, or a Riemannian manifold if you prefer) we could rewrite the definition of metric distortion in a suggestive way. Namely, if $\phi : X \to \mathbb{R}^n$, the metric distortion is,

$$\delta_0(\phi) = \sup_{z \in Z_0 X} \frac{\mathrm{Fill}_{\mathcal{H}}\, \phi_* z}{\mathrm{Fill}_X\, z} \cdot \sup_{z \in Z_0 X} \frac{\mathrm{Fill}_X\, z}{\mathrm{Fill}_{\mathcal{H}}\, \phi_* z}$$

The 0-cycles here are taken with $\mathbb{Z}_2$ coefficients and the associated norm, the L^1-norm. The function $\mathrm{Fill}(z)$ is defined as $\mathrm{Fill}(z) = \inf\{\|y\| : \partial y = z\}$.

The filling function defines a common metric on the space of cycles, referred to as the *flat metric* (or flat *norm*) [26],

$$\|z - w\|_\flat = \mathrm{Fill}(z - w).$$

This metric has been extensively studied (see [14] for a modern treatment). From this perspective then, we can regard the filling distortion as the *metric distortion* of the induced maps on flat cycles, i.e.

$$\delta_k(X \xrightarrow{\phi} Y) = \delta_0(Z_k X \xrightarrow{\phi_*} Z_k Y)$$

This was our main motivation for defining filling distortion in the way that we did.

5.2. **The volume-respecting embeddings of U. Feige.** There is another noteworthy generalization of the concept of metric distortion. Feige [9] defined the notion of volume-respecting embeddings; let us define it here.

Definition (Feige, [9])**.** If (S,d) is a finite metric space, the *volume* of S is defined as the supremum of the volume of the convex hull of the image of S under a 1-Lipschitz map from S to $\mathbb{R}^{|S|-1}$. More formally,

$$\mathrm{Vol}(S) := \sup_{\phi,\, 1\text{-Lipschitz}} \mathrm{vol}_{|S|-1}\left[\mathrm{convex}(\phi(s_1),\ldots,\phi(s_{|S|}))\right].$$

Now, let (X,d) be a finite metric space endowed with an additional hypergraph structure, $\chi \in 2^X$ (we can take, for example, all subsets of X of size less than M). Then for a 1-Lipschitz map $\Phi : X \to \mathcal{H}$ the *volume distortion* of Φ is defined as

$$\eta(\Phi) := \max_{S\in\chi}\left[\frac{\mathrm{Vol}(S)}{\mathrm{vol}_{|S|-1}\mathrm{convex}(\Phi(S))}\right]^{\frac{1}{|S|-1}}$$

where $\mathrm{Vol}(S)$ is the volume of S as a metric space in its own right.

Example. Let (X,d) be a finite metric space and let χ be the hypergraph structure consisting of all pairs of points in X, then for any map $\phi : X \to \mathcal{H}$,

$$\eta(\phi) = \delta_0(\phi).$$

So Feige's volume distortion is indeed a generalization of metric distortion.

Volume distortion, however, is distinct from filling distortion as we have defined it. For example, every simplicial complex on n vertices with complete 1-skeleton (and any hypergraph structure desired) can be embedded by some ϕ into $\mathbb{R}^n$ with $\eta(\phi) = 1$. This shows, in particular, the dependence of volume distortion on the underlying metric.

5.3. **Filling distortion and the fundamental group.** In the last few years there has been some innovative work of the topology of Linial–Meshulam complexes. The work of Babson, Hoffman and Kahle is a prime example.

Theorem (Babson, Hoffman, Kahle, [4])**.** *Let $\omega(n)$ be any function of n which tends to infinity as n does. If $p > \left(\frac{3\log n - \omega(n)}{n}\right)^{\frac{1}{2}}$, then with probability tending to 1 as $n \to \infty$, $Y \in Y_{n,p}$ has $\pi_1(Y) = 0$.*

For any $\delta > 0$, if $p < n^{\frac{1}{2}-\delta}$, then asymptotically almost surely, $Y \in Y_{n,p}$ has $\pi_1(Y) \neq 0$.

Now if we reexamine our main theorem, we notice that our estimates on filling distortion break down when, $\epsilon = \frac{1}{2}$. The reason for this is explained in [4]: The fundamental group of Y vanishes at the same threshold that every triangle, τ, is the boundary of a disk with a *bounded* number of faces. Therefore, our lower bound on $\sum_\tau (\mathrm{Fill}_X \tau)^2$ must degenerate at the threshold $p = n^{\frac{1}{2}}$ and we can no longer obtain a sensible bound on filling distortion. This phenomenon is real because,

$$\inf_\phi \delta_1(\phi) \leq \max_\tau \mathrm{Fill}_X \tau$$

(just by embedding the vertices of the complex as the standard basis of $\mathbb{R}^n$).

5.4. **Higher dimensions.** We have chosen to state all of the theorems and propositions in this article in terms of 2-dimensional complexes. We felt that writing all arguments in their generality was cumbersome and of little use to the reader. However, we would like to reassure the reader (or the reader can reassure themselves) that every argument can be made to apply to $(k+1)$-dimensional complexes with simple modifications. All of these modifications are straightforward with the possible exception of the definition of ψ in the proof of proposition 1.3, so we provide it here.

In general, use the function, $\psi : X^{(k)} \to \mathbb{R}^{\binom{n}{k+1}}$ defined by

$$\begin{aligned} \psi_{(i_0 < \cdots < i_k)}(\sigma) &= \int_{\phi\sigma} \sum_j (-1)^j x_{i_j} dx_{i_0} \wedge \cdots \wedge \hat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k} \\ &\quad + \int_{\phi\sigma} \sum_{(j_1 < \cdots < j_k)} y_{(j_1 < \cdots < j_k)}^{(i_0 < \cdots < i_k)} dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \end{aligned}$$

In light of this cumbersome formula, it may occur to the reader now why we decided to omit the general case. With all these considerations, theorem 1.2 comes out to:

Theorem. *For every large n and every ϵ, there exists a $(k+1)$-dimensional simplicial complex on n vertices and complete 1-skeleton with the property that every affine map $\phi : X \to \mathcal{H}$ has,*

$$\delta_1(\phi) \geq C n^{\frac{k-(k+1)\epsilon}{(k+1)^2}}$$

5.5. **Manifolds.** If one emulates the proof [20] that a k-regular graph, G, (with its shortest distance metric) requires $C(k)\sqrt{\lambda^0}\log|G|$ metric distortion to embed into Euclidean space, we immediately see that the proof extends to manifolds:

Proposition 5.1. *Let (M^n, g) be a hyperbolic manifold whose Laplacian (on functions) has spectral gap $\lambda^0(M, g)$. Then any map $\phi : M \to \mathcal{H}$ has metric distortion,*

$$\delta^0(\phi) \geq C\sqrt{\lambda^0} \log \mathrm{vol}_n(M, g).$$

It seems natural then, to ask if there is an estimate,

$$\delta^k(\phi) \geq F(\lambda^k(M, g), \mathrm{vol}_n(M, g)),$$

for every map, ϕ, from a hyperbolic manifold into $\mathcal{H}$.

5.6. **Heilbronn triangle problem.** A problem dating back to the late 1940's asks, "Arranging n points in a unit disk, how big can we make the area of every triangle [24]?" More specifically, for a given arrangement, X, of n points in a disk, consider all the triangles, τ, formed among these points and define

$$H(n) := \sup_X \min_\tau \mathrm{Area}(\tau).$$

The problem of estimating $H(n)$ is commonly called the *Heilbronn triangle problem.* The problem has stubbornly resisted a complete solution despite the attention of many suitors (see [17] for a short history). The current bounds, which have stood since the early 1980's [16] [17], are,

$$C\frac{\log n}{n^2} \leq H(n) \leq Kn^{-\frac{8}{7}-\epsilon}.$$

Now, we can always choose a simplicial complex (with n vertices and complete 1-skeleton) to our liking *after* a configuration of points has been chosen (e.g. including all the faces which are empty triangles [1] or some other scheme). Provided we are able to estimate the spectral gap $\lambda^1(X)$ and $\sum_\tau (\mathrm{Fill}_X \tau)^2$, can our distortion estimates shed any light on the Heilbronn problem?

5.7. **Extremal complexes.** Given the estimate in proposition 1.3, it seems a natural question to ask: What simplicial complexes (on n vertices and complete 1-skeleton) maximize the quantity:

$$\frac{\lambda^1(X)}{|X^{(2)}|} \sum_\tau \Big(\mathrm{Fill}_X(\tau)\Big)^2 \qquad ?$$

The author has no idea how to systematically optimize this quantity, but the optimizers may be very interesting.

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