Topology of random 2-complexes

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June 11, 2010

Abstract

We study the Linial–Meshulam model of random two-dimensional simplicial complexes. One of our main results states that for $p \ll n^{-1}$ a random 2-complex Y collapses simplicially to a graph and, in particular, the fundamental group $\pi_1(Y)$ is free and $H_2(Y) = 0$, a.a.s. We also prove that, if the probability parameter p satisfies $p \gg n^{-1/2+\epsilon}$, where $\epsilon > 0$, then an arbitrary finite two-dimensional simplicial complex admits a topological embedding into a random 2-complex, with probability tending to one as $n \to \infty$. We also establish several related results, for example we show that for $p < c/n$ with $c < 3$ the fundamental group of a random 2-complex contains a nonabelian free subgroup. Our method is based on exploiting explicit thresholds (established in the paper) for the existence of simplicial embedding and immersions of 2-complexes into a random 2-complex.

1 Introduction

Modeling of large systems in applications motivates the development of unconventional geometric and topological notions. Among them are mixed probabilistic - topological concepts, such as the Erdös and Rényi random graphs of [ER60], which are currently used in many applications in engineering and computer science.

More recently, higher dimensional analogs of the Erdős-Rényi model were suggested and studied by Linial-Meshulam in [LM06], and Meshulam-Wallach in [MW09]. In these models one generates a random d-dimensional complex Y by considering the full d-dimensional skeleton of the simplex Δ_n on vertices $\{1, \dots, n\}$ and retaining d-dimensional faces independently with probability p.

An interesting class of closed smooth manifolds depending on a large number of random parameters arise as configuration spaces of mechanical linkages with bars of random lengths, see [Far08], [FK]. Although the number of homeomorphism type of these manifolds grows extremely fast, their topological characteristics can be predicted with high probability when the number of links tends to infinity.

In this paper, we study the topology random two-dimensional complexes. The probability space $G(\Delta_n^{(2)}, p)$ of the Linial–Meshulam model of random 2-complexes is defined as follows. Let Δ_n denote the $(n-1)$-dimensional simplex with vertices $\{1, 2, \dots, n\}$. Then $G(\Delta_n^{(2)}, p)$ denotes the set of all 2-dimensional subcomplexes

$$\Delta_n^{(1)} \subset Y \subset \Delta_n^{(2)},$$

*Partly supported by a grant from the EPSRC.
†Partly supported by the Swiss National Science Foundation.

containing the one-dimensional skeleton $\Delta_n^{(1)}$. The probability function $\mathbb{P} : G(\Delta_n^{(2)}, p) \to \mathbf{R}$ is given by the formula

$$\mathbb{P}(Y) = p^{f(Y)}(1-p)^{\binom{n}{3}-f(Y)}, \quad Y \in G(\Delta_n^{(2)}, p),$$

where $f(Y)$ denotes the number of faces in Y. In other words, each of the 2-dimensional simplexes of $\Delta_n^{(2)}$ is included in a random 2-complex Y with probability p, independently of the other 2-simplexes. As in the case of random graphs, $0 < p < 1$ is a probability parameter which may depend on n. The model $G(\Delta_n^{(2)}, p)$ includes all finite 2-dimensional simplicial complexes containing the full 1-skeleton $\Delta_n^{(1)}$; however, the likelihood of various topological phenomena is dependent on the value of p. The theory of deterministic 2-complexes itself is a rich and active field of current research with many challenging open questions, see [HMS93].

The fundamental group of a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ was investigated by Babson, Hoffman, and Kahle [BHK08]. They showed that for

$$p \gg n^{-1/2} \cdot (3 \log n)^{1/2},$$

the group $\pi_1(Y)$ vanishes asymptotically almost surely (a.a.s)[1]. These authors use notions of negative curvature due to Gromov to study the nontriviality and hyperbolicity of $\pi_1(Y)$ for

$$p \ll n^{-1/2-\epsilon}.$$

In [CFK10] it was shown that for $p \ll n^{-1-\epsilon}$, a random 2-complex Y can be collapsed to a graph in N steps, where $N = N(\epsilon)$ depends only on $\epsilon > 0$.

In this paper we prove the following theorem:

Theorem 1. *If the probability parameter p satisfies*

$$p \ll n^{-1}$$

then a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ collapses simplicially to a graph, a.a.s. In particular, the fundamental group $\pi_1(Y)$ is free and for any coefficient group G one has $H_2(Y; G) = 0$, a.a.s.

We conjecture that a similar result holds for d-dimensional random complexes in the Meshulam - Wallach model [MW09], i.e. for $p \ll n^{-1}$ a random d-dimensional complex collapses simplicially to a $(d-1)$-dimensional subcomplex. This would strengthen a theorem of D. Kozlov [Koz09].

Another major result of this paper states:

Theorem 2. *Assume that for some $\epsilon > 0$ the probability parameter p satisfies $p \gg n^{-1/2+\epsilon}$. Let S be an arbitrary simplicial finite 2-complex. Then S admits a topological embedding into a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$, a.a.s.*

By *a topological embedding $S \to Y$* we mean a simplicial embedding of a subdivision of S into Y.

The method of this paper (as well as the method of [CFK10]) is based on studying simplicial embeddings and immersions of polyhedra into random 2-complexes. We analyze in detail the numerical invariants $\mu(S)$ and $\tilde{\mu}(S)$, defined in section §3, which play a crucial role in the questions about the existence of embeddings and immersions. We also discuss the notion of balanced triangulations, a generalization of the notion of a balanced graph in the random graph theory. We prove that any triangulation of a closed surface is balanced although surfaces with boundary (even disks) admit unbalanced triangulations.

Among some other results presented in this paper we may mention the statement that for $p < c/n$, where $c < 3$, the fundamental group of a random 2-complex contains a nonabelian free subgroup, a.a.s. We also prove that for $p > c/n$ with $c > 3$ the second homology group of a random 2-complex is nontrivial a.a.s; this strengthens a result of D. Kozlov [Koz09].

[1] We use the abbreviation a.a.s. for the phrase "asymptotically almost surely".

Basic Definitions

For convenience of the reader we collect here the definitions of basic combinatorial notions related to 2-dimensional complexes which will be used in this paper.

Let Y be a finite 2-dimensional simplicial complex. An edge of Y is called *free* if it is included in exactly one 2-simplex.

The *boundary* ∂Y is defined as the union of all free edges. We say that a 2-complex Y is *closed* if $\partial Y = \emptyset$.

A 2-complex Y is called *pure* if every maximal simplex is 2-dimensional. By the *pure part* of a 2-complex we mean the maximal pure subcomplex, i.e. the union of all 2-simplexes.

Let Y be a simplicial 2-complex and let σ and τ be two 2-simplexes of Y. We say that σ and τ are adjacent if they intersect in an edge. The *distance* between σ and τ, $d_Y(\sigma, \tau)$, is the minimal integer k such that there exists a sequence of 2-simplexes $\sigma = \sigma_0, \sigma_1, \dots, \sigma_k = \tau$ with the property that σ_i is adjacent to σ_{i+1} for every $0 \le i < k$. (If no such sequence exists then $d_Y(\sigma, \tau) = \infty$.) The *diameter* $\mathrm{diam}(Y)$ is defined as the maximal value of $d_Y(\sigma, \tau)$ taken over pairs of 2-simplexes of Y.

A simplicial 2-complex is *strongly connected* if it has a finite diameter.

A *pseudo-surface* is a finite, pure, strongly connected 2-dimensional simplicial complex of degree at most 2 (i.e., every edge is included in at most two 2-simplexes).

2 The fundamental group and the second Betti number

In this section we analyze the fundamental group and the second Betti number of a random 2-complex using mainly information provided by the Euler characteristic. The results of this section are specific for 2-dimensional random complexes.

Theorem 3. *Suppose that $p < cn^{-1}$, where $c < 3$. Then the fundamental group $\pi_1(Y)$ of a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ contains a noncommutative free subgroup with probability at least $1 - \lambda^{n^2}$, for all large enough n, where*

$$\lambda = \exp\left(-\frac{1}{8}\left(1 - \frac{c}{3}\right)^2\right),$$

$0 < \lambda < 1$. *In particular, $\pi_1(Y)$ contains a free subgroup on two generators, a.a.s.*

Proof. The Euler characteristic of $Y \in G(\Delta_n^{(2)}, p)$ can be written as

$$\chi(Y) = n - \binom{n}{2} + f_2(Y) = f_2(Y) + 1 - \binom{n-1}{2} \tag{1}$$

where $f_2(Y)$ denotes the number of 2-simplexes in Y. Clearly, the function $f_2 : G(\Delta_n^{(2)}, p) \to \mathbf{Z}$ coincides with the sum of random variables

$$f_2 = \sum_\sigma I_\sigma$$

where σ runs over 2-simplexes (i, j, k) (with $1 \le i < j < k \le n$) and $I_\sigma(Y) = 1$ iff σ is included in Y; otherwise $I_\sigma(Y) = 0$. Each I_σ is a Bernoulli random variable with parameter p and f_2 has binomial distribution

$$\mathbb{P}(f_2(Y) = k) = \binom{\binom{n}{3}}{k} p^k (1-p)^{\binom{n}{3} - k},$$

where $k = 0, 1, 2, \ldots, \binom{n}{3}$. The expectation $\mathbb{E}(f_2)$ equals $p\binom{n}{3}$. Using inequality (2.5) from [JŁR00] we find that for any $t \geq 0$

$$\mathbb{P}\left(f_2 \geq p\binom{n}{3} + t\right) \leq \exp\left(-\frac{t^2}{2(p\binom{n}{3} + t/3)}\right). \tag{2}$$

Consider inequality (2) with

$$t = \left(1 - \frac{pn}{3}\right)\binom{n-1}{2} - 1. \tag{3}$$

We observe that: (i) the assumption $pn < c < 3$ implies that $t > 0$ for large n and (ii) the inequality $f_2(Y) \geq p\binom{n}{3} + t$ is equivalent to the inequality $\chi(Y) \geq 0$. We thus obtain from (2)

$$\mathbb{P}\left(\chi(Y) \geq 0\right) \leq \exp\left(-\frac{t^2}{2(p\binom{n}{3} + t/3)}\right)$$

and from (3), for $n \geq 3$,

$$p\binom{n}{3} + \frac{t}{3} \leq \frac{1}{3}\left(\frac{2c}{3} + 1\right)\binom{n-1}{2} - \frac{1}{3} \leq \binom{n-1}{2}$$

as $c < 3$. Thus one gets for n sufficiently large

$$\begin{aligned} \frac{t^2}{2(p\binom{n}{3} + t/3)} &\geq \frac{1}{2}\frac{[(1-\frac{pn}{3})\binom{n-1}{2} - 1]^2}{\binom{n-1}{2}} \\ &\geq \frac{1}{3}\left(1 - \frac{c}{3}\right)^2 \cdot \binom{n-1}{2} \\ &\geq \frac{1}{8}\left(1 - \frac{c}{3}\right)^2 \cdot n^2. \end{aligned}$$

Therefore, by the definition of λ,

$$\mathbb{P}(\chi(Y) \geq 0) \leq \exp\left(-\frac{1}{8}\left(1 - \frac{c}{3}\right)^2 n^2\right) = \lambda^{n^2}$$

and thus

$$\mathbb{P}(\chi(Y) < 0) \geq 1 - \lambda^{n^2}.$$

Theorem 3 now follows from a theorem proven in [FS06] which states: *If the Euler characteristic of a finite connected two-dimensional polyhedron Y is negative, $\chi(Y) < 0$, then $\pi_1(Y)$ contains a nonabelian free subgroup.*

This completes the proof.

$\square$

Theorem 4. *Suppose that $p > cn^{-1}$, where now $c > 3$. Then for a random two-dimensional complex $Y \in G(\Delta_n^{(2)}, p)$ one has $H_2(Y; \mathbf{Z}) \neq 0$ with probability at least $1 - \mu^{n^2}$, for all large enough n, where*

$$\mu = \exp\left(-\frac{1}{8}\left(\frac{c}{3} - 1\right)\right),$$

$0 < \mu < 1$. *In particular*[2], $H_2(Y; \mathbf{Z}) \neq 0$, *a.a.s.*

[2] Note that $H_2(Y; \mathbf{Z}) \neq 0$ implies that $H_2(Y; G) \neq 0$ for any coefficient group G.

Proof. The proof is very similar to the one of Theorem 3 and also uses the Euler characteristic. Clearly, $\chi(Y) = 1 - b_1(Y) + b_2(Y)$ (where $b_i(Y)$ denotes the i-dimensional Betti number, $b_i(Y) = \mathsf{rk} H_i(Y; \mathbf{Z})$). Thus $\chi(Y) > 1$ implies $b_2(Y) > 0$. We will estimate from above the probability of the complementary event $\chi(Y) \leq 1$.

Using inequality (2.6) from [JŁR00] one has for any $t \geq 0$

$$\mathbb{P}\left(f_2 \leq p\binom{n}{3} - t\right) \leq \exp\left(-\frac{t^2}{2p\binom{n}{3}}\right).$$

Now choose

$$t = \left(\frac{pn}{3} - 1\right) \cdot \binom{n-1}{2}.$$

Since $pn > c > 3$ we have

$$t > \left(\frac{c}{3} - 1\right) \cdot \binom{n-1}{2} > 0.$$

The inequality $f_2(Y) \leq p\binom{n}{3} - t$ is equivalent to $\chi(Y) \leq 1$. Thus we obtain

$$\mathbb{P}\left(\chi(Y) \leq 1\right) \leq \exp\left(-\frac{t^2}{2p\binom{n}{3}}\right)$$

and, for n sufficiently large,

$$\begin{aligned} \frac{t^2}{2p\binom{n}{3}} &\geq \frac{(\frac{pn}{3} - 1)^2 \cdot \binom{n-1}{2}}{2\frac{pn}{3}} \\ &\geq \frac{1}{2}\left(\frac{pn}{3} - 1\right) \cdot \binom{n-1}{2} \\ &\geq \frac{1}{8}\left(\frac{c}{3} - 1\right) \cdot n^2. \end{aligned}$$

Finally, by the definition of μ,

$$\mathbb{P}(b_2(Y) = 0) \;\leq\; \mathbb{P}(\chi(Y) \leq 1) \;\leq\; \mu^{n^2}.$$

This completes the proof. □

Next we consider the critical case $p = 3/n$.

Theorem 5. *Assume that $p = \frac{3}{n}$. Then for any $\epsilon > 0$ there exists N such that for all $n > N$ the probability of each of the following statements (a) and (b) concerning a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ is greater than $\frac{1}{2} - \epsilon$:*

(a) the fundamental group $\pi_1(Y)$ contains a noncommutative free subgroup;

(b) $H_2(Y; \mathbf{Z}) \neq 0$.

It is not known if (a) and (b) exclude each other; one may ask about the probability that asymptotically, (a) and (b) hold simultaneously.

Proof. In the case when $p = 3/n$ one has $\mathbb{E}(f_2) = \binom{n-1}{2}$ and $\mathbb{E}(\chi) = 1$ where $f_2, \chi : G(\Delta_n^{(2)}, p) \to \mathbf{Z}$ are as above. From the De Moivre-Laplace Integral theorem [Sh96], page 62, it follows that

$$\mathbb{P}\left(f_2 > \binom{n-1}{2}\right) \sim \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dx = \frac{1}{2}$$

and

$$\mathbb{P}\left(f_2 \leq \binom{n-1}{2} - 2\right) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-x^2/2} dx \sim \frac{1}{2},$$

where b can be found from the equation $b\sqrt{\binom{n}{3}p(1-p)} = -2$, i.e.

$$b = -\frac{2}{\sqrt{\binom{n-1}{2} \cdot \left(1 - \frac{3}{n}\right)}} \sim 0.$$

By (1), the inequality $f_2(Y) > \binom{n-1}{2}$ is equivalent to $\chi(Y) > 1$ and the inequality $f_2(Y) < \binom{n-1}{2} - 1$ is equivalent to $\chi(Y) < 0$. Thus we see that

$$\mathbb{P}(\chi(Y) > 1) \sim \frac{1}{2}, \quad \text{and} \quad \mathbb{P}(\chi(Y) < 0) \sim \frac{1}{2}$$

and thus, for any given $\epsilon > 0$,

$$\mathbb{P}(\pi_1(Y) \supset F_2) \geq \mathbb{P}(\chi(Y) < 0) \geq \frac{1}{2} - \epsilon,$$

$$\mathbb{P}(b_2(Y) > 0) \geq \mathbb{P}(\chi(Y) > 1) \geq \frac{1}{2} - \epsilon$$

for sufficiently large n. Here F_2 denotes the free group with two generators. □

3 Simplicial embeddings and immersions

In this section we consider the containment problem for subcomplexes of random 2-dimensional complexes which is similar to the containment problem for random graphs, see [JŁR00], chapter 3. We also study simplicial immersions, which are more general than simplicial embeddings.

Let S be a 2-dimensional finite simplicial complex. We denote by $v = v_S$ and $f = f_S$ the numbers of vertices and faces of S respectively. The set of vertices of S is denoted by $V(S)$. We assume that S is fixed, i.e. independent of n.

Definition 6. *A simplicial embedding $g : S \hookrightarrow Y$, where $Y \in G(\Delta_n^{(2)}, p)$ is a random 2-complex, is defined as an injective map of the set of vertices $V(S)$ of S into the set of vertices $\{1, \dots, n\}$ of Y satisfying the following condition: for any triple of distinct vertices $u_1, u_2, u_3 \in V(S)$ which span a simplex in S, the corresponding points $g(u_1), g(u_2), g(u_3) \in \{1, \dots, n\}$ span a face of Y.*

Next we define the following slightly more general notion.

Definition 7. *A simplicial immersion $g : S \looparrowright Y$ into a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ is defined as a map of the set of vertices $V(S)$ of S into the set of vertices $\{1, \dots, n\}$ of Y satisfying the following two conditions:*

(a) for any triple of distinct vertices $u_1, u_2, u_3 \in V(S)$ which span a 2-simplex in S, the corresponding points $g(u_1), g(u_2), g(u_3) \in \{1, \dots, n\}$ are pairwise distinct and span a face of Y;

(b) for any pair of distinct 2-simplexes σ and σ' of S, the corresponding 2-simplexes $g(\sigma)$ and $g(\sigma')$ of Y are distinct.

Note that a simplicial immersion $g : S \looparrowright Y$ is not necessarily injective on the set of vertices $V(S)$ but any pair of vertices $u_1, u_2 \in V(S)$ with $g(u_1) = g(u_2)$ cannot lie in a 2-simplex of S. We also require that distinct 2-simplexes of S are mapped to distinct 2-simplexes of Y.

If $g : S \looparrowright Y$ is a simplicial immersion then for any subcomplex $S' \subset S$ the restriction $g|S'$ is also a simplicial immersion $S' \looparrowright Y$.

Lemma 8. *The probability that a 2-dimensional simplicial complex S with v vertices and f faces admits a simplicial immersion into a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ is less or equal than $n^v p^f$, i.e.*

$$\mathbb{P}(S \looparrowright Y) \leq n^v p^f. \tag{4}$$

Proof. For a map $g : V(S) \to \{1, \dots, n\}$ denote by $J_g : G(\Delta_n^{(2)}, p) \to \{0, 1\}$ the random variable such that $J_g(Y) = 1$ if and only if g determines a simplicial immersion $S \looparrowright Y$, i.e. if the condition of Definition 7 is satisfies. Clearly, the expectation $\mathbb{E}(J_g)$ equals p^f. The random variable $X_S = \sum_g J_g$ counts the number of simplicial immersions $S \looparrowright Y$, where g runs over all maps $V(S) \to \{1, \dots, n\}$. Thus

$$\mathbb{E}(X_S) = \sum_g \mathbb{E}(J_g) \leq n^v \cdot p^f$$

and

$$\mathbb{P}(S \looparrowright Y) = \mathbb{P}(X_S > 0) \leq \mathbb{E}(X_S) \leq n^v p^f,$$

by the first moment method. □

Next we define a useful numerical invariant which was also mentioned in [BHK08].

Definition 9. *For a simplicial 2-complex S let $\mu(S)$ denote*

$$\mu(S) = \frac{v}{f} \in \mathbf{Q},$$

where $v = v_S$ and $f = f_S$ are the numbers of vertices and faces in S.

Corollary 10. *If the probability parameter p satisfies*

$$p \ll n^{-\mu(S)}$$

then the 2-complex S admits no simplicial immersions into a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$, a.a.s.

Proof. The assumption $p \ll n^{-\mu(S)}$ means that $pn^{\mu(S)} \to 0$ as $n \to \infty$. Then $n^v p^f \to 0$ and the result now follows from Lemma 8. □

As an example consider a simplicial graph Γ and the cone over it $S = C(\Gamma)$. One has $v_S = v_\Gamma + 1$ and $f_S = e_\Gamma$. Therefore

$$\mu(S) = \frac{v_\Gamma + 1}{e_\Gamma}.$$

Using Corollary 10 we obtain:

Corollary 11. *If a graph Γ satisfies $\chi(\Gamma) < 0$ then $\mu(S) \leq 1$ where $S = C(\Gamma)$ is the cone over Γ. Therefore, if, $p \ll n^{-1}$, then the cone $S = C(\Gamma)$ with $\chi(\Gamma) < 0$ admits no simplicial immersions into a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$, a.a.s.*

This result will be used later in this paper.

Definition 12. *Let S be a finite 2-dimensional simplicial complex. Define*

$$\tilde{\mu}(S) = \min_{S' \subset S} \mu(S'), \tag{5}$$

where the minimum is formed over all subcomplexes $S' \subset S$ or, equivalently, over all pure subcomplexes $S' \subset S$.

Note that the invariant $\tilde{\mu}$ is monotone decreasing: if S is a subcomplex of T then $\tilde{\mu}(S) \geq \tilde{\mu}(T)$.

The following result complements Corollary 10.

Theorem 13. *Let S be a finite simplicial complex.*

(A) If $p \ll n^{-\tilde{\mu}(S)}$ then the probability that S admits a simplicial immersion into a random 2-complex $Y \subset G(n,p)$ tends to zero as $n \to \infty$.

(B) If $p \gg n^{-\tilde{\mu}(S)}$ then the probability that S admits a simplicial embedding into a random 2-complex $Y \subset G(n,p)$ tends to one as $n \to \infty$.

Proof. Let $S' \subset S$ be a subcomplex such that $\mu(S') = \tilde{\mu}(S) \leq \mu(S)$. Then

$$\mathbb{P}(S \looparrowright Y) \leq \mathbb{P}(S' \looparrowright Y)$$

and $\mathbb{P}(S' \looparrowright Y)$ tends to zero assuming that $p \ll n^{-\mu(S')} = n^{-\tilde{\mu}(S)}$ by Corollary 10. This proves the statement (A).

The following arguments prove the statement (B). Let v denote the number of vertices of S. A simplicial embedding of S into Y is defined by an injective map $g : V(S) \to \{1, \dots, n\}$ where $V(S)$ is the set of vertices of S. The function $X_S = \sum_g J_g : G(\Delta_n^{(2)}, p) \to \mathbf{Z}$ counts the number of simplicial embeddings; here $g : V(S) \to \{1, \dots, n\}$ runs over all injective maps and J_g denotes the random variable defined as in the proof of Lemma 8.

For a pair of injective maps $g, g' : V(S) \to \{1, \dots, n\}$ consider the pure subcomplex $H = H(g, g') \subset S$ which is defined as the union of all 2-simplexes $\sigma \subset S$ with the property $g(\sigma) \subset g'(S)$. Note that the product random variable $J_g J_{g'}$ has the expectation

$$\mathbb{E}(J_g J_{g'}) = p^{2f - f_H},$$

where $f = f_S$ is the number of faces of S and f_H is the number of 2-simplexes in H.

Now we fix a pure subcomplex $H \subset S$ and consider all ordered pairs of injective maps $g, g' : V(S) \to \{1, \dots, n\}$ with $H(g, g') = H$. The number N of such pairs g, g' satisfies

$$N \leq C_H n^{2v - v_H}$$

for some constant $C_H > 0$ depending on H.

The variance of X_S can be estimated as follows

$$\begin{aligned}
\mathrm{Var}(X_S) &= \mathbb{E}(X_S^2) - \mathbb{E}(X_S)^2 \\
&= \sum_{g,g'} [\mathbb{E}(J_g J_{g'}) - \mathbb{E}(J_g)\mathbb{E}(J_{g'})] \\
&\leq \sum_{H \subset S} C_H n^{2v - v_H} \left[p^{2f - f_H} - p^{2f}\right] \\
&= \sum_{H \subset S} C_H n^{2v - v_H} p^{2f - f_H} [1 - p^{f_H}].
\end{aligned}$$

Since for n sufficiently large,

$$\mathbb{E}(X_S) = \binom{n}{v} v! \cdot p^f \;\geq\; \frac{1}{2} \cdot n^v p^f,$$

it follows that

$$\frac{\mathrm{Var}(X_S)}{\mathbb{E}(X_S)^2} \leq 4 \cdot (1-p) \cdot \sum_{H \subset S} (f_H C_H) \cdot (n^{v_H} p^{f_H})^{-1}.$$

Now, if $p \gg n^{-\tilde{\mu}(S)}$ then $n^{v_H} p^{f_H} \to \infty$ for any pure subcomplex $H \subset S$ and therefore each term in the sum above tends to zero. Thus using the Chebyshev inequality

$$\mathbb{P}(X_S = 0) \leq \frac{\mathrm{Var}(X_S)}{\mathbb{E}(X_S)^2},$$

we see that $\mathbb{P}(X_S = 0) \to 0$ as $n \to \infty$. This implies statement (B). □

The above proof gives also the following quantitative statement:

Corollary 14. *Let S be a fixed 2-complex. Then the probability $\mathbb{P}(S \not\subset Y)$ that S is not embeddable into a random 2-complex Y can be estimated by*

$$\mathbb{P}(S \not\subset Y) \leq C \cdot (1-p) \cdot \sum_{H \subset S, f_H > 0} (n^{v_H} p^{f_H})^{-1}, \tag{6}$$

where C is a constant depending on S and H runs through all pure subcomplexes of S.

4 Proof of Theorem 1.

In this section we prove Theorem 1 stated in the Introduction.

Note that the assumptions and conclusions of Theorem 1 are stronger than those of Theorem 3. One may also compare Theorem 1 with the main result of [CFK10] which has stronger assumptions and conclusion than Theorem 1.

Proof. For any triple of integers x, y, z (with $x \geq 3$, $y \geq 3$ and $z \geq 0$) consider two graphs $\Gamma_{x,y,z}$ and $\Gamma'_{x,y,z}$ drawn schematically in Figure 1. The graph $\Gamma_{x,y,z}$ is topologically the union of two

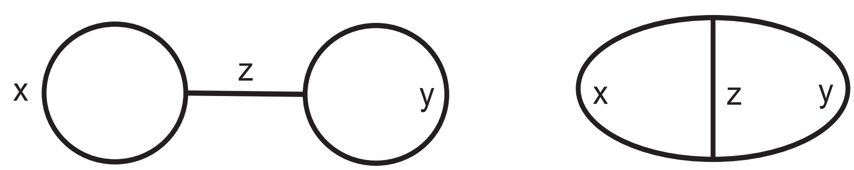

Figure 1: Graphs $\Gamma_{x,y,z}$ (left) and $\Gamma'_{x,y,z}$ (right).

circles joined by an interval; the circle on the left consists of x intervals, the circle on the right is subdivided into y intervals, and the interval connecting them consists of z subintervals. The graph $\Gamma'_{x,y,z}$ shown schematically on the right of Figure 1, is the union of three arcs consisting of x, y and z intervals. Clearly $\chi(\Gamma_{x,y,z}) = -1 = \chi(\Gamma'_{x,y,z})$. In the case $z = 0$ the corresponding interval degenerates to a point.

It is easy to see that *any graph Γ with $\chi(\Gamma) < 0$ contains, as a subgraph, either $\Gamma_{x,y,z}$, or $\Gamma'_{x,y,z}$, for some x, y, z.*

Consider the cones $S_{x,y,z} = C(\Gamma_{x,y,z})$ and $S'_{x,y,z} = C(\Gamma'_{x,y,z})$. By the arguments leading to Corollary 11 we have

$$\mu(S_{x,y,z}) = \mu(S'_{x,y,z}) = 1.$$

Applying Lemma 8 we find

$$\mathbb{P}(S_{x,y,z} \not\hookrightarrow Y) \leq (pn)^f$$

where $f = x + y + z$. Thus,

$$\sum_{x,y\geq 3,\, z\geq 0} \mathbb{P}(S_{x,y,z} \looparrowright Y) \leq \sum_{f\geq 6} f^2 \cdot (pn)^f$$

$$\leq \sum_{f\geq 6} (2pn)^f = \frac{(2pn)^6}{1-2pn}.$$

We see that *if $pn \to 0$, then the probability that there exist x, y, z such that the 2-complex $S_{x,y,z}$ admits a simplicial immersion into Y tends to zero as $n \to \infty$.*

Similarly, *if $pn \to 0$, then the probability that there exist x, y, z such that the 2-complex $S'_{x,y,z}$ admits a simplicial immersion into Y tends to zero.*

Consider a vertex v of the random 2-complex Y. The link L_v of v is a graph and the cone $C(L_v)$ embeds simplicially into Y. If for a connected component L'_v of L_v one has $\chi(L'_v) < 0$ then for some integers x, y, z the component L'_v contains either $\Gamma_{x,y,z}$ or $\Gamma'_{x,y,z}$. Thus we see that $\chi(L'_v) < 0$ implies that for some x, y, z the complex Y contains either $S_{x,y,z}$ or $S'_{x,y,z}$ as a subcomplex. Using the arguments given above we obtain that *for any vertex v of Y, the Euler characteristic of every connected component L'_v of the link L_v of v satisfies*

$$\chi(L'_v) \geq 0,$$

a.a.s. In other words, every connected component of the link of any vertex of Y is either contractible or is homotopy equivalent to the circle.

Let S be a pure and closed simplicial subcomplex of Y. The above arguments show that the link of any vertex of S is a disjoint union of circles. In other words, we obtain that *any pure closed subcomplex $S \subset Y$ is a closed pseudo-surface*, i.e. every edge of S is incident to exactly two 2-simplexes of S, a.a.s.

For any two positive integers $x, y \geq 3$ with $\max(x, y) \geq 4$, let $L_{x,y}$ be a subdivision of the disk D^2 shown in Figure 2. The complex $L_{x,y}$ has two internal vertices v, w such that the degree of v is x and the degree of w is y. The total number of vertices of $L_{x,y}$ equals $x + y - 2$; the number

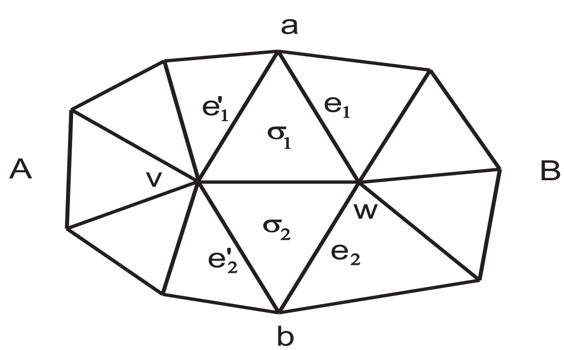

Figure 2: 2-complex $L_{x,y}$.

of faces of $L_{x,y}$ is also $x + y - 2$; therefore $\mu(L_{x,y}) = 1$.

In the special case $x = 3$ and $y = 3$ the complex $L_{3,3}$ is defined to be the tetrahedron with vertices v, w, a, b. The equality $\mu(L_{3,3}) = 1$ remains true.

By Lemma 8,

$$\mathbb{P}(L_{x,y} \looparrowright Y) \leq (pn)^f,$$

where $f = x + y - 2$. Thus,

$$\sum_{x,y\geq 3} \mathbb{P}(L_{x,y} \looparrowright Y) \leq \sum_{f\geq 4} f\cdot (pn)^f \leq \sum_{f\geq 4} (2pn)^f = \frac{(2pn)^4}{1-(2pn)}.$$

This shows that, if $pn \to 0$, then, with probability tending to one as $n\to\infty$, none of the complexes $L_{x,y}$ can be immersed $L_{x,y} \looparrowright Y$ into Y.

Next we show that for any nonempty closed pseudo-surface S there exist positive integers $x, y \geq 3$ and an immersion $L_{x,y} \looparrowright S$. Consider an edge $e = vw$ of S and two 2-simplexes σ_1 and σ_2 incident to it, as shown on Figure 2. The link of v in S is a disjoint union of circles. It contains the edges e_1 and e_2 shown on Figure 2. Therefore we may find a simple arc A in the link of v in S connecting the points a and b and disjoint from the interior of the arc $e_1 \cup e_2$. Similarly we may find a simple arc B connecting a and b in the link of w in S and disjoint from the interior of $e_1' \cup e_2'$. Let x and y be such that the number of 2-simplexes in arc A (correspondingly, B) is $x - 2$ (correspondingly $y - 2$). It is now obvious that we obtain *an immersion* of the 2-complex $L_{x,y}$ into S. It may not be an embedding since the images of some points of A may coincide with the images of some points of B.

Now we see that if a random 2-complex contains a closed pseudo-surface then there is an immersion $L_{x,y} \looparrowright Y$.

Hence, summarizing the statements made above, we conclude that *in the case $pn \to 0$, a random 2-complex contains no nonempty closed two-dimensional subcomplexes $S \subset Y$, a.a.s.*

Let Y be a finite simplicial 2-complex. An edge of Y is called *free* if it is incident to a single 2-simplex. A 2-simplex of Y is called *free* if at least one of its edges is free. Let $\sigma_1, \dots, \sigma_k$ be all free 2-simplexes of Y; pick a sequence of free edges $e_1, \dots, e_k$ with $e_i \subset \sigma_i$. The subcomplex

$$Y' = Y - \cup_{i=1}^k \mathrm{int}(\sigma_i) - \cup_{i=1}^k \mathrm{int}(e_i)$$

is obtained from Y by collapsing all free 2-simplexes. The operation $Y \searrow Y'$ is called *a simplicial collapse*. Clearly, $Y' \subset Y$ is a deformation retract of Y.

The procedure of collapse can be iterated $Y \searrow Y' \searrow Y'' \searrow \dots$. There are two possibilities: either (a) after a finite number of collapses we obtain a *closed* 2-dimensional complex $Y^{(k)}$; or (b) for some k the complex $Y^{(k)}$ is one-dimensional, i.e. a graph.

Our discussion above implies that if $pn \to 0$, then for a random 2-complex Y the possibility (a) happens with probability tending to 0. Therefore, with probability tending to 1, a random 2-complex collapses to a graph, under the assumption $p \ll n^{-1}$.

This completes the proof. □

Remark 15. The main step of the above proof was to show that for $p \ll n^{-1}$ a random 2-complex Y contains no nonempty closed 2-dimensional subcomplexes $S \subset Y$. From Lemma 19 of [CFK10] we know that for any closed 2-complex S one has $\tilde{\mu}(S) \leq 1$. Therefore, given a closed 2-complex S, we may apply Theorem 13 to conclude that the probability that this S embeds into a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ tends to zero as $n \to \infty$. However this would not be strong enough to prove Theorem 1 since we need to know (as shown in the proof above) that the probability that there exists a closed 2-complex S which embeds to a random 2-complex tends to zero.

5 Surfaces in random 2-complexes

In this section we apply the results of section §3 and study embeddings of triangulated surfaces into random 2-dimensional complexes.

Definition 16. *A finite simplicial 2-complex S is called balanced if*

$$\mu(S) = \tilde{\mu}(S),$$

i.e. if the quantities defined in Definitions 9 and 12 coincide. In other words, S is balanced if

$$\mu(S) \leq \mu(S')$$

for any subcomplex $S' \subset S$.

Definition 16 is similar to the corresponding notion for random graphs, see [JŁR00].

In this section we show that there exist many unbalanced triangulations of the disk however all closed triangulated surfaces are balanced. We start with the following observation.

Lemma 17. *A connected simplicial 2-complex S is balanced if and only if $\mu(S) \leq \mu(S')$ for all connected subcomplexes $S' \subset S$.*

Proof. Let $S' = S'_1 \sqcup S'_2$ be a disjoint union of two subcomplexes. We show that

$$\mu(S') \geq \min\{\mu(S'_1), \mu(S'_2)\}$$

and thus $\mu(S) \leq \mu(S'_i)$, where $i = 1, 2$, implies $\mu(S) \leq \mu(S')$. Let v_i and f_i denote the number of vertices and faces of S'_i, $i = 1, 2$. Assume that $v_1/f_1 \leq v_2/f_2$. Then one easily checks

$$\mu(S') = \frac{v_1 + v_2}{f_1 + f_2} \geq \frac{v_1}{f_1} = \mu(S'_1).$$

The result now follows by induction on the number of connected components of S'. □

Example 18. Let $S = \Sigma_g$ be a triangulated closed orientable surface of genus $g \geq 0$. Then $\chi(S) = 2 - 2g = v - e + f$ where v, e, f denote the numbers of vertices, edges and faces in S correspondingly. Each edge is contained in two faces which gives $3f = 2e$ and therefore

$$\mu(\Sigma_g) = \frac{1}{2} + \frac{2 - 2g}{f}. \tag{7}$$

Similarly, if $S = N_g$ is a triangulated closed nonorientable surface of genus $g \geq 1$ then $\chi(N_g) = 2 - g$ and

$$\mu(N_g) = \frac{1}{2} + \frac{2 - g}{f}. \tag{8}$$

Formulae (7) and (8) give the following:

Corollary 19. *The invariants $\mu(\Sigma_g)$ of orientable triangulated surfaces satisfy:*

1. $1/2 < \mu(\Sigma_g) \leq 1$ *for $g = 0$ (since $f \geq 4$);*
2. $\mu(\Sigma_g) = 1/2$ *for $g = 1$ (the torus);*
3. $\mu(\Sigma_g) < 1/2$ *for $g > 1$;*
4. *If $f \to \infty$ (i.e. when the surface is subsequently subdivided) then $\mu(\Sigma_g) \to 1/2$.*

Corollary 20. *The invariants $\mu(N_g)$ of nonorientable triangulated surfaces satisfy:*

1. $1/2 < \mu(N_g) \leq 3/5$ *for $g = 1$ (since $f \geq 10$);*
2. $\mu(N_g) = 1/2$ *for $g = 2$ (the Klein bottle);*
3. $\mu(N_g) < 1/2$ *for $g > 2$;*

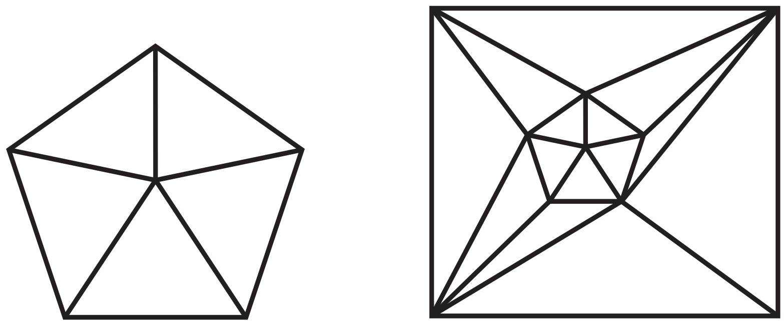

Figure 3: An n-gon S (left) and a square with implanted n-gon T (right).

4. If $f \to \infty$ (i.e. when the surface is subsequently subdivided) then $\mu(N_g) \to 1/2$.

Here we used the well-known fact that any triangulation of the real projective plane $\mathbf{RP}^2$ has $f \geq 10$ faces, see [Hea90], [JR80], [HR91], [Rin55].

Example 21. Let S be a triangulated disc. Then $\chi(S) = v - e + f = 1$ and $3f = 2e - e_0$ where e_0 is the number of edges in the boundary ∂S. Substituting $e = (3f + e_0)/2$, one obtains

$$\mu(S) = \frac{1}{2} + \frac{e_0}{2f} + \frac{1}{f}. \tag{9}$$

As a specific example consider the regular n-gon S shown on Figure 3 left. Then $v = n + 1$, $f = n$ and

$$\mu(S) = 1 + \frac{1}{n}.$$

On Figure 3 on the right we have $e_0 = 4$ and the number of faces f equals $f = 2n + 4$. Thus

$$\mu(T) = \frac{1}{2} + \frac{3}{2n+4},$$

converges to $\frac{1}{2}$ as $n \to \infty$.

Corollary 22. *For any triangulation S of the disk one has $\mu(S) > 1/2$. There exist triangulations S of D^2 with $\mu(S)$ arbitrarily close to $1/2$.*

Example 23. Let S' be such that $\mu(S') < 1$ and suppose that S is obtained from S' by adding a triangle Δ such that $S' \cap \Delta$ is an edge. Then S is not balanced. Indeed, $v_S = v_{S'} + 1$ and $f_S = f_{S'} + 1$ and

$$\mu(S) = \frac{v_{S'} + 1}{f_{S'} + 1} > \frac{v_{S'}}{f_{S'}} = \mu(S').$$

Corollary 24. *There exist unbalanced triangulations of the disk.*

Proof. Start with a disk triangulation S' with $\mu(S') < 1$ (for instance, S' can be the square with implanted n-gon, see Example 21) and add a triangle $S = S' \cup \Delta$ such that $S' \cap \Delta$ is an edge lying in the boundary $\partial S'$. Then $\mu(S) > \mu(S')$ (see Example 23) and S is unbalanced. Clearly, S is homeomorphic to the 2-dimensional disk. $\square$

Theorem 25. *Any closed connected triangulated surface S is balanced.*

Proof. Let $S' \subset S$ be a connected subcomplex, $S' \neq S$. We may assume that each edge of S' belongs to either one or two triangles of S' (since any edge which is not incident to a triangle can be simply removed without affecting $\mu(S')$). Then we have

$$\chi(S') = 1 - b_1(S') = v' - e' + f' \tag{10}$$

where v', e', f' are the numbers of vertices, edges and faces in S'. Here we use the assumption that S' is connected (i.e. $b_0(S') = 1$) and $S' \neq S$ (i.e. $b_2(S') = 0$). One may write

$$3f' = 2e' - e_0$$

where e_0 is the number of edges incident to exactly one 2-simplex. Expressing e' through f', and e_0 and substituting into (10) we obtain

$$\mu(S') = \frac{1}{2} + \frac{1 - b_1(S')}{f'} + \frac{e_0}{2f'}. \tag{11}$$

Assume first that S is orientable and has genus g, i.e. $S = \Sigma_g$. Then we have formula (7) and the inequality $\mu(S') \geq \mu(S)$ is equivalent to

$$\frac{1 - b_1(S')}{f'} + \frac{e_0}{2f'} \quad \geq \quad \frac{2 - 2g}{f}$$

or

$$f[2 - 2b_1(S') + e_0] \quad \geq \quad (4 - 4g)f',$$

where f denotes the number of 2-simplexes in S. Since $f \geq f'$ the above inequality follows from

$$2 - 2b_1(S') + e_0 \geq 4 - 4g.$$

Since $b_1(S) = 2g$ the latter inequality is equivalent to

$$b_1(S') \leq b_1(S) + e_0/2 - 1. \tag{12}$$

The homological exact sequence of (S, S') has the form

$$\begin{aligned} 0 \to H_2(S; \mathbf{Q}) \quad &\to \quad H_2(S, S'; \mathbf{Q}) \stackrel{j_*}{\to} H_1(S'; \mathbf{Q}) \\ &\to \quad H_1(S; \mathbf{Q}) \to H_1(S, S'; \mathbf{Q}) \to 0. \end{aligned}$$

Here $H_2(S; \mathbf{Q}) = \mathbf{Q}$ and by the Poincaré duality theorem (see [Hat02], Proposition 3.46)

$$H_2(S, S'; \mathbf{Q}) \simeq H^0(S - S'; \mathbf{Q}) \tag{13}$$

has dimension equal to the number k of path-connected components of the complement $S - S'$. Formally, we find a compact deformation retract $K \subset S - S'$ such that $S - K$ deformation retracts onto S' and apply Proposition 3.46 from [Hat02] to it; thus we obtain (13).

It follows that the image of j_* has dimension $k - 1$ and therefore the long exact sequence implies

$$b_1(S) \geq b_1(S') - k + 1. \tag{14}$$

Each of the connected components of the complement $S - S'$ is bounded by a simple polygonal curve having at least 3 edges. Therefore, we see that

$$e_0 \geq 3k \tag{15}$$

and now (12) follows from (14).

Consider now the case when the surface S is nonorientable, $S = N_g$. In this case the arguments are similar but we will consider the homology groups with coefficients in $\mathbf{Z}_2$ and the $\mathbf{Z}_2$-Betti numbers which we will denote

$$b_i'(X) = \dim H_i(X; \mathbf{Z}_2).$$

Comparing $\mu(S')$ given by (11) and $\mu(S)$ given by (8) and taking into account the equality

$$b_1'(S) = g,$$

we see that the inequality $\mu(S') \geq \mu(S)$ is equivalent to

$$b_1'(S') \leq b_1'(S) + e_0/2 - 1, \tag{16}$$

which is analogous to (12). The inequality (16) follows from arguments similar to the ones given above with $\mathbf{Z}_2$ coefficients replacing the rationals $\mathbf{Q}$, using the Poincaré duality and the inequality (15). □

In the following statement we consider *"small surfaces"*, i.e. triangulated surfaces which do not depend on n. Theorems 13, 25 and Corollaries 19 and 20 imply:

Corollary 26. *One has:*

1. *If $p \ll n^{-1}$ then a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ contains*[3] *no small*[4] *closed surfaces, a.a.s.*
2. *If $n^{-1} \ll p \ll n^{-3/5}$ then a random 2-complex Y contains small spheres but no small closed surfaces of other topological types, a.a.s.*
3. *If $n^{-3/5} \ll p \ll n^{-1/2}$ then a random 2-complex Y contains small spheres and projective planes but no small closed surfaces of higher genera, a.a.s.*
4. *If $p \gg n^{-1/2}$ then a random 2-complex Y contains all small spheres, projective planes, tori and Klein bottles, a.a.s.*
5. *If $p \gg n^{-1/2+\epsilon}$ for some $\epsilon > 0$ then, given a topological type of a closed surface, there exists $f_0 = f_0(\epsilon)$, such that any triangulation of the surface having more than f_0 2-simplexes will be simplicially embeddable into a random 2-complex Y, a.a.s. In particular, if $p \gg n^{-1/2+\epsilon}$, a random 2-complex Y contains small closed orientable and nonorientable surfaces of all possible topological types, a.a.s.*

Proof. These statements follow from Theorem 25 and formulae (7) and (8). □

The statement 5 of the previous Corollary can be compared with Theorem 2 which deals with topological embeddings.

Corollary 27. *For a random 2-complex $Y \in G(\Delta_n^{(2)}, p)$ with $p \gg n^{-1}$ one has*

$$\pi_2(Y) \neq 0, \quad \textit{and} \quad H_2(Y; \mathbf{Z}) \neq 0 \tag{17}$$

a.a.s.

Proof. Indeed, by the previous Corollary, for $p \ll n^{-1}$ a random 2-complex Y contains a tetrahedron as a simplicial subcomplex. The fundamental class of this tetrahedron gives a nontrivial element of $H_2(Y)$. The tetrahedron can also be viewed as a sphere in Y representing a nontrivial class in $\pi_2(Y)$. □

The statement $H_2(Y; \mathbf{Z}) \neq 0$ also follows from Theorem 4 and from the result of D. Kozlov [Koz09].

[3] In this Corollary the word "contains" means "contains as a simplicial subcomplex."

[4] In this statement one may remove the word "small" as follows from the proof of Theorem 1.

6 Remarks concerning the invariant $\mu(S)$

First we observe that $\mu(S)$ admits the following curious interpretation.

For each vertex $u_i \in V(S)$ its degree $\deg(u_i)$ is defined as the number of edges incident to u_i. For an edge $e_i \in E(S)$ the degree $\deg(e_i)$ is defined as the number of two-dimensional simplexes incident to e_i. Next we define *the average vertex degree* and *the average edge degree* by the formulae

$$D_v(S) = v^{-1} \cdot \sum_{u_i \in V(S)} \deg(u_i), \quad D_e(S) = e^{-1} \cdot \sum_{e_i \in E(S)} \deg(e_i).$$

Lemma 28. *For any 2-complex S one has*

$$\mu(S) \cdot D_v(S) \cdot D_e(S) = 6.$$

Proof. The statement follows from the definition

$$\mu(S) = v/f = 6 \cdot \frac{v}{2e} \cdot \frac{e}{3f}$$

using the following obvious formulae

$$3f = \sum_{e_i \in E(S)} \deg(e_i), \quad 2e = \sum_{u_i \in V(S)} \deg(u_i).$$

□

Lemma 29. *For any strongly connected 2-complex S one has*

$$\mu(S) \leq 1 + \frac{2}{f}, \tag{18}$$

where $f = f_S$ is the number of faces in S.

Proof. Without loss of generality we may assume that S is pure; otherwise we apply the arguments below to the pure part of S.

Given a pure strongly connected 2-complex S, there exists a sequence of subcomplexes $T_1 \subset T_2 \subset \cdots \subset T_f = S$ such that (a) each T_i has exactly i faces, i.e. $f_{T_i} = i$, and (b) the subcomplex T_{i+1} is obtained from T_i by adding a single 2-simplex σ_i with the property that the intersection $\sigma_i \cap T_i$ contains an edge of σ_i. If v_i denotes the number of vertices of T_i then $v_{i+1} \leq v_i + 1$. Since $v_1 = 3$, it follows that $v = v_f \leq f + 2$ implying (18). □

Corollary 30. *Suppose that a 2-complex $S = S_1 \cup S_2$ is the union of two strongly connected subcomplexes such that the intersection $S_1 \cap S_2$ is at most one-dimensional. (a) If $S_1 \cap S_2$ contains at least 4 vertices then $\mu(S) \leq 1$. (b) If the intersection $S_1 \cap S_2$ contains ≥ 5 vertices then $\mu(S) < 1$.*

Proof. Denote $v_i = v_{S_i}$, $f_i = f_{S_i}$, where $i = 1, 2$ and, as usual, $v = v_S$, $f = f_S$. By the previous Lemma, $v_i \leq f_i + 2$, and thus we obtain

$$\begin{aligned} \mu(S) &= \frac{v_1 + v_2 - v_0}{f_1 + f_2} \leq \frac{f_1 + 2 + f_2 + 2 - v_0}{f_1 + f_2} && (19) \\ &= 1 + \frac{4 - v_0}{f_1 + f_2}, && (20) \end{aligned}$$

where v_0 is the number of vertices lying in the intersection $S_1 \cap S_2$. Thus, $\mu(S) \leq 1$ if $v_0 \geq 4$ and $\mu(S) < 1$ if $v_0 > 4$. □

Lemma 31. *Let S be a connected, pure, closed (i.e. $\partial S = \emptyset$) 2-complex with $\chi(S) = 1$ having at least 3 edges of degree ≥ 3. Then*

$$\mu(S) \leq \frac{1}{2} - \frac{1}{2f}, \tag{21}$$

where $f = f_S$ is the number of faces.

Proof. We have

$$v - e + f = 1 \tag{22}$$

(since $\chi(S) = 1$) and

$$3f \geq 2e + 3. \tag{23}$$

The last inequality follows from the formula

$$3f = 2e + e_3 + e_4 + \dots$$

where e_r denotes the number of edges of degree at least r in S with $r = 3, 4, \dots$. From (22) and (23) we obtain $v \leq \frac{f}{2} - \frac{1}{2}$ implying (21). □

An example of a 2-complex satisfying the condition of the previous Lemma is the house with two rooms, see [Hat02], page 4.

7 Topological embeddings: proof of Theorem 2

Proof. We show that there exists a subdivision of S which simplicially embeds into Y a.a.s.

We subdivide S by introducing a new vertex in the center of each 2-simplex and connecting it to three vertices, as shown on Figure 4. We denote by S' the new triangulation. Let v, f and

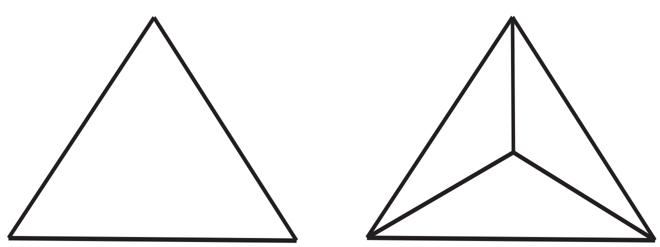

Figure 4: A 2-simplex (left) and its subdivision (right).

v', f' denote the numbers of vertices and faces of S and S' respectively. Then clearly

$$v' = v + f, \quad f' = 3f.$$

Therefore we find that

$$\mu(S') - \frac{1}{2} = \frac{1}{3}\left(\mu(S) - \frac{1}{2}\right). \tag{24}$$

We claim that a similar formula holds for $\tilde{\mu}$, i.e.

$$\tilde{\mu}(S') - \frac{1}{2} = \frac{1}{3}\left(\tilde{\mu}(S) - \frac{1}{2}\right). \tag{25}$$

Indeed, let $T \subset S$ be a subcomplex. Then its subdivision T' (defined as explained above) is a subcomplex of S', and the numbers $\mu(T)$ and $\mu(T')$ are related by the equation (24). We show below that

$$\tilde{\mu}(S') = \min_{T\subset S} \mu(T'). \tag{26}$$

Clearly, (26) implies

$$\tilde{\mu}(S') = \min_{T\subset S}\left[\frac{1}{3}\left(\mu(T) - \frac{1}{2}\right) + \frac{1}{2}\right] = \frac{1}{3}\left(\tilde{\mu}(S) - \frac{1}{2}\right) + \frac{1}{2}$$

which is equivalent to (25).

To prove the formula (26) consider a subcomplex $R \subset S'$. Each 2-simplex σ of S determines three 2-simplexes of S' which we denote by $\sigma_1, \sigma_2, \sigma_3$. We want to show that we may replace R by a subcomplex $R_1 \subset S'$ such that $\mu(R_1) \leq \mu(R)$ and either R_1 contains all simplexes $\sigma_1, \sigma_2, \sigma_3$ or it contains none of them.

Suppose that R contains σ_1 and σ_2 but does not contain σ_3. Then $R_1 = R \cup \sigma_3$ has the same number of vertices and greater number of faces, i.e. $\mu(R_1) < \mu(R)$.

Suppose now that R contains only one simplex among the σ_i's; assume, that, say, $\sigma_1 \subset R$ and $\sigma_2 \not\subset R$ and $\sigma_3 \not\subset R$. (A) If $\mu(R) \geq 1/2$, define R_1 by $R_1 = R \cup \sigma_2 \cup \sigma_3$. Then $\mu(R_1) \leq \mu(R)$. (B) If $\mu(R) \leq 1$ define R_1 as R with σ_1 removed; then $\mu(R_1) \leq \mu(R)$. Clearly at least one of the cases (A) or (B) holds and we proceed by induction, repeating this procedure with respect to all 2-simplexes $\sigma \subset S$. Thus we see that the minimum in

$$\tilde{\mu}(S') = \min_{R\subset S'} \mu(R)$$

is achieved on subcomplexes $R \subset S'$ which have the form $R = T'$ for some $T \subset S$. This completes the proof of (25).

For $r = 0, 1, 2, \dots$ denote by S^r the simplicial 2-complex which is obtained from S by r consecutive subdivisions as above. Then from (25) we obtain

$$\tilde{\mu}(S^r) - \frac{1}{2} = \frac{1}{3^r}\left(\tilde{\mu}(S) - \frac{1}{2}\right). \tag{27}$$

We see that this sequence approaches 1/2 as $r \to \infty$. It follows that, given $\epsilon > 0$, for all sufficiently large r we have

$$\tilde{\mu}(S^r) \geq 1/2 - \epsilon.$$

Thus, the assumption $p \gg n^{-1/2+\epsilon}$ implies $p \gg n^{-\tilde{\mu}(S^r)}$ and now we may apply Theorem 13 to conclude that the r-th subdivision S^r simplicially embeds into Y, a.a.s. Hence we see that S topologically embeds into Y, a.a.s. □

Remark 32. The result of Theorem 2 cannot be improved (without adding extra hypothesis) despite a special type of subdivision used in the proof. Indeed, one sees from formulae (7) and (8) and Theorem 25 that for a closed orientable surface Σ_g of genus $g \geq 1$ one has $\tilde{\mu}(\Sigma_g) \to 1/2$ as the number of 2-simplexes f goes to infinity. A similar conclusion is valid for nonorientable surfaces N_g with $g \geq 2$.

Remark 33. Consider the following invariant $\mathrm{sign}(X) \in \{+1, -1, 0\}$ of a simplicial 2-complex:

$$\mathrm{sign}(X) = \mathrm{sign}\left(\tilde{\mu}(X) - \frac{1}{2}\right).$$

Formula (24) seems to suggest that it is topologically invariant. However in (24) we used a special type of subdivisions. The following example shows that *in general* $\mathrm{sign}(X)$ *is not topologically invariant.* Consider the 2-complex X shown in Figure 5 (left) which is the union of three triangles having a common edge. Let Y_k be obtained by adding k new vertices along the common edge and connecting them to the remaining vertices, see Figure 5 (right). One has $\tilde{\mu}(X) = 5/3$ and

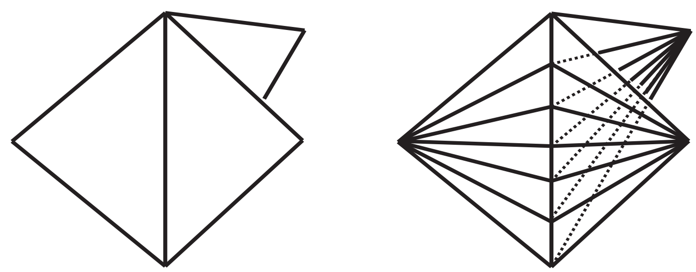

Figure 5: Complex X (left) and its subdivision Y_k (right).

therefore $\mathrm{sign}(X) = +1$. However

$$\tilde{\mu}(Y_k) \le \mu(Y_k) = \frac{k+5}{3k+3}.$$

Thus, for $k > 7$, one has $\tilde{\mu}(Y_k) < 1/2$ and $\mathrm{sign}(Y_k) = -1$.

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