Beyond random graphs : random simplicial complexes. Applications to sensor networks

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Plan

Sensor networks

A wireless sensor network (WSN) is a wireless network consisting of spatially distributed autonomous devices using sensors to cooperatively monitor physical or environmental conditions.

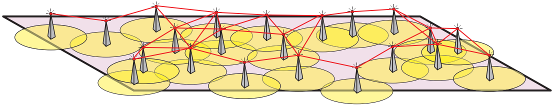
Wikipedia

A sensor

A sensor is defined by

1. position
2. coverage radius

at each time.



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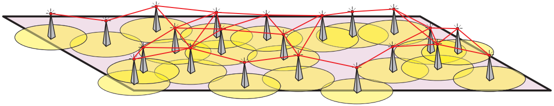

Some questions

All positions known : Domain covered ?

Some positions known : Optimal locations of other sensors

Positions varying with time : Creation of holes ?

Fault-tolerance/ Power-saving : Can we support failure (or switch-off) without creating holes ?

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Mathematical framework

Homology : Algebraization of the topology

Coverage : reduces to compute the rank of a matrix

Detection of hole, redundancy : reduces to the computation of a basis of a vector matrix, obtained by matrix reduction (as in Gauss algorithm).

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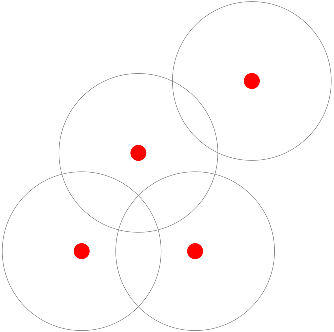
Cech complex

$$C_k = \bigcup \{[x_0, \cdots, x_{k-1}],\ x_i \in \omega, \cap_{i=0}^{k} B(x_i, \epsilon) \neq \emptyset\}$$

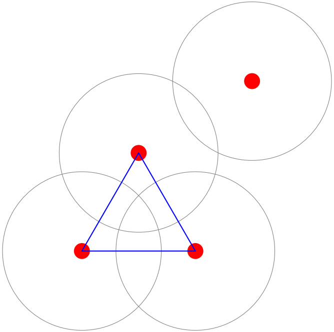
Nerve theorem

The set $\bigcup_{x \in \omega} B(x, \epsilon)$ has the same homology groups as the simplicial complex $(C_k,\ k \geq 0)$.

Rips complex



Rips complex



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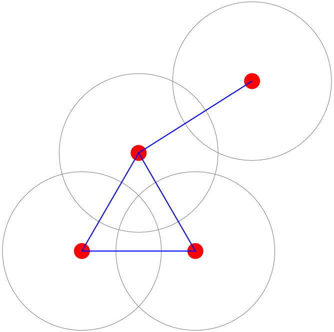

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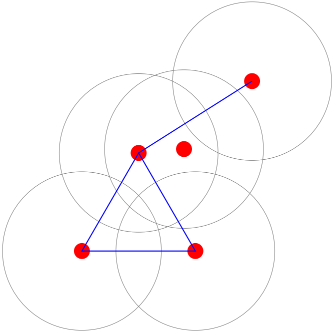

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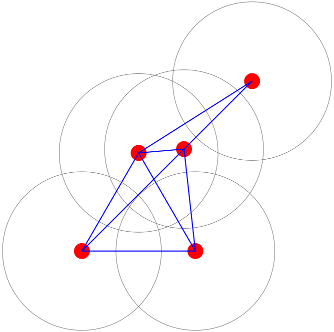

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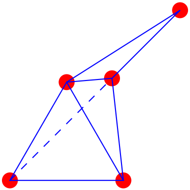

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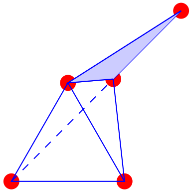
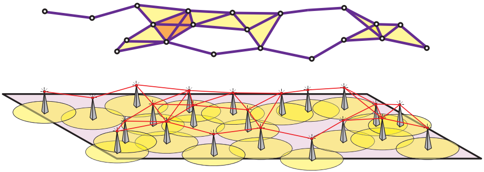


Rips complex of sensor network (cf. [dSG07, dSG06, GM05])



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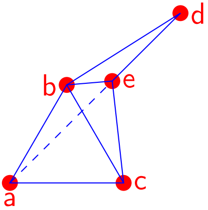


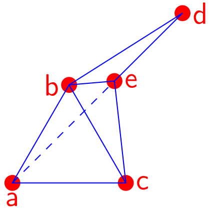
Vertices : a, b, c, d, e.

Edges : ab, bc, ca, ae, be, ec, bd, ed.

Triangles : abc, abe, bec, aec, bed.

Tetrahedron : abec.

Rips complex



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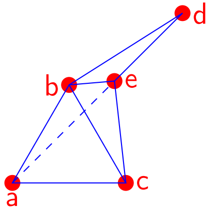

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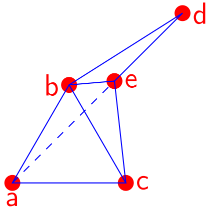

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Rips and Cech

- No hole in Cech implies no hole in Rips complex.
- The converse does not hold.
- For l^{∞} distance, Rips=Cech.

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Simplices algebra [Hat02, ZC05]

- $ab = -ba$
- $3ab$ means three times the edge ab.

Boundary operator

$$\partial_{n-1}\ a_1 a_2 \dots a_n = \sum_{i=1}^{n} (-1)^i a_1 a_2 \dots \widehat{a_i} \dots a_n.$$

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Example :

$$\partial_2\ abe = be - ae + ab.$$

Main result

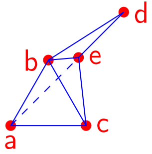
Main observation

$$\partial_n \partial_{n+1} = 0$$

$$\begin{aligned}\partial_1 \partial_2 \; abe &= \partial_1(be - ae + ab) \\ &= e - b - (e - a) + b - a = 0.\end{aligned}$$

Cycles and boundaries

- A triangle is a cycle of edges.
- A tetrahedron is a cycle of triangles.
- A triangle is a boundary of a tetrahedron.



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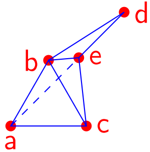

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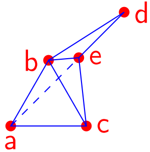

Betti numbers

β_0=number of connected components

$$\beta_0 = \text{ Nb of vertices} - \text{ Nb of } \textit{independent} \text{ edges.}$$

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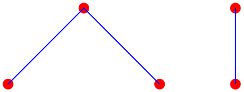

$$\beta_0 = 5 - 3 = 2.$$

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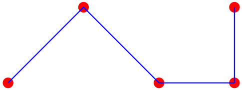

$$\beta_0 = 5 - 4 = 1.$$

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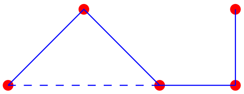

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β_1=number of « holes »

$\beta_1 =$ Nb of *independent* polygons − Nb of independent triangles.

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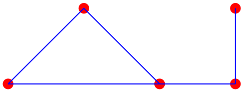

$$\beta_1 = 1 - 1 = 0.$$

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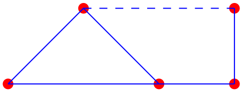

$$\beta_1 = 2 - 1 = 1.$$

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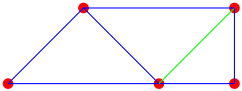

$$\beta_1 = 3 - 3 = 0.$$

For larger n

- $\beta_n = \dim \ker \partial_n - \dim \text{range } \partial_{n+1}$.
- Hard to visualize the intuitive meaning
- But relatively easy to compute.
- Existence of tetrahedron in Rips complex means over-coverage.

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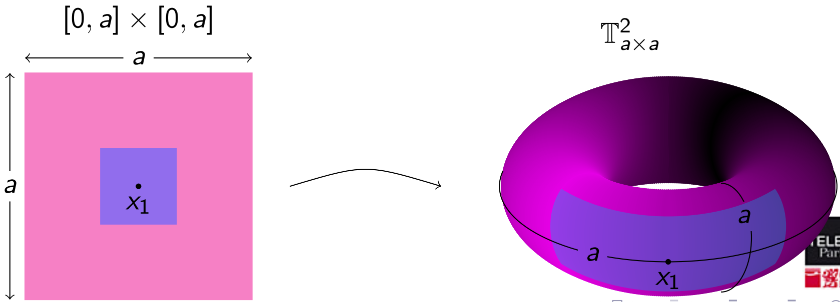
Random setting

- Sensors = Poisson process (λ)
- Domain = d dimensional torus of width a
- Coverage = square of width ε
- r-dilation of P.P.(λ)=P.P.(λr^{-d}), one can choose $a = 1$.



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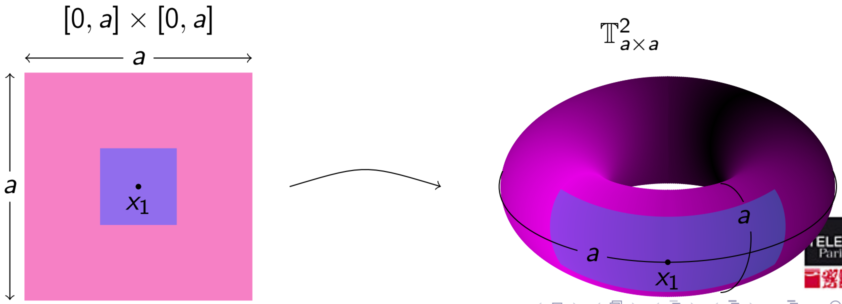

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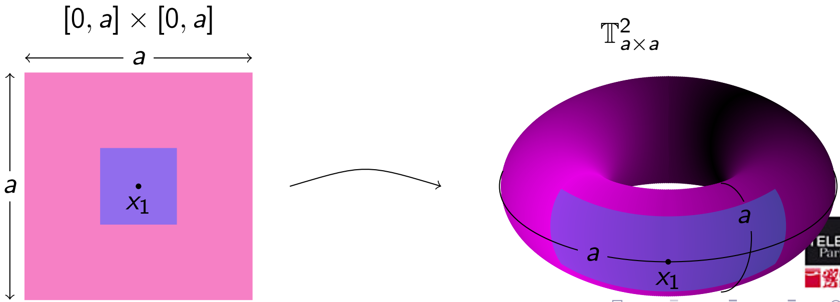

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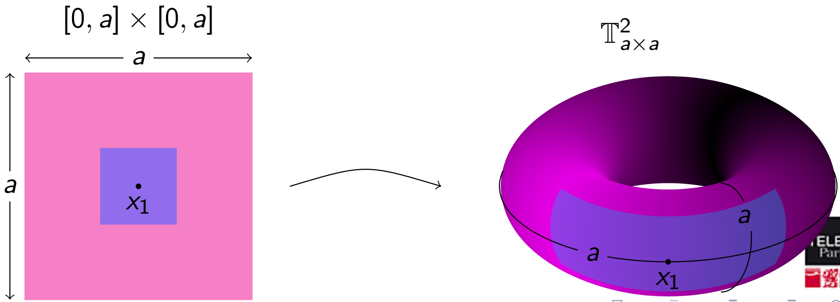

Euler characteristic

$$\chi = \sum_{k=0}^{d} (-1)^k \beta_k.$$

- d=1 : $\{\chi = 0 \cap \beta_0 \neq 0\} \Leftrightarrow$ { circle is covered }
- d=2 : $\{\chi = 0 \cap \beta_0 \neq \beta_1\} \Leftrightarrow$ { domain is covered }
- d=3 : $\{\chi = 0 \cap \beta_0 + \beta_2 \neq \beta_1\} \Leftrightarrow$ { space is covered }

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$$\chi = \sum_{k=1}^{\infty} (-1)^k s_k .$$

Euler characteristic

- $B_d(x)$: Bell polynomial

$$B_d(x) = \left\{ \begin{matrix} d \\ 1 \end{matrix} \right\} x + \left\{ \begin{matrix} d \\ 2 \end{matrix} \right\} x^2 + \ldots + \left\{ \begin{matrix} d \\ d \end{matrix} \right\} x^d$$

Euler characteristic

$$\mathbf{E}\left[\chi\right] = -\frac{\lambda e^{-\theta}}{\theta} B_d(-\theta) \text{ where } \theta = \lambda \epsilon^d.$$

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k simplices

- Define $h(x_1, \dots, x_k) \triangleq \frac{1}{k!}\mathbb{I}_{\{\|x_i - x_j\| < \epsilon, i \neq j\}}(x_1, \dots, x_k)$
- Then (Campbell) :

$$\overline{s_k} = \lambda^{k+1} \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} h(x_1, \dots, x_{k+1}) dx_{k+1} \dots dx_1$$
$$= \frac{(k+1)^d}{(k+1)!} \lambda \theta^k$$

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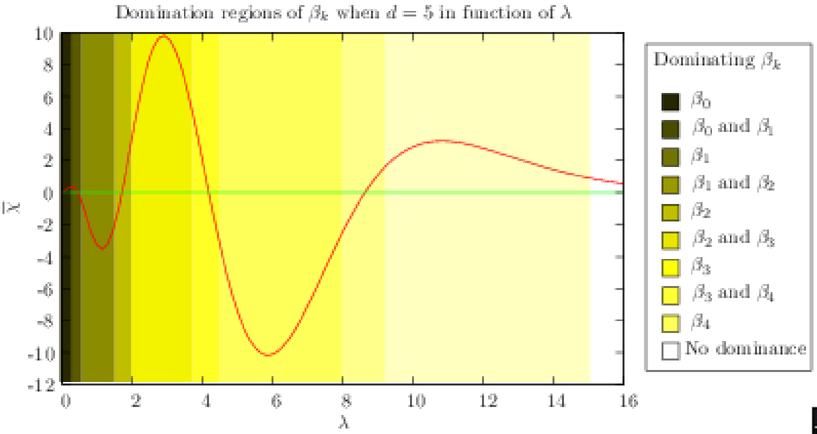
Dimension 5


Domination regions of β_k when $d = 5$ in function of λ
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
$\bar{\chi}$
0
2
4
6
8
10
12
14
16
λ
Dominating β_k
β_0
β_0 and β_1
β_1
β_1 and β_2
β_2
β_2 and β_3
β_3
β_3 and β_4
β_4
No dominance

Second order moments

$$\mathrm{Cov}(s_k,\, s_l) = \left(\frac{1}{2\epsilon}\right)^d \sum_{i=0}^{l-1} \frac{1}{i!(k-l+i)!(l-i)!}\theta^{k+i} \times \left(k+i+2\frac{i(k-l+i)}{l-i+1}\right)^d .$$

Euler characteristic

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$$\mathrm{Var}(\chi) = \left(\frac{1}{d}\right)^d \sum_{i=1}^{\infty} c_i \theta^i$$

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In dimension 1,

$$\mathrm{Var}(\chi) = \left(\theta e^{-\theta} - 2\theta^2 e^{-2\theta}\right)$$

Asymptotic results

If $\lambda \to \infty$, $\beta_i(\omega) \xrightarrow{p.s.} \beta_i(\mathbb{T}^d) = \binom{d}{i}$.

Limit theorems

CLT for Euler characteristic

$$\text{distance}_{TV}\left(\frac{\chi - \mathbf{E}[\chi]}{\sqrt{V_\chi}},\ \mathfrak{N}(0,1)\right) \le \frac{c}{\sqrt{\lambda}}.$$

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- Stein method
- Malliavin calculus for Poisson process

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Concentration inequality

- Discrete gradient $D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$
- $D_x \beta_0 \in \{1, 0, -1, -2, -3\}$

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$c > \mathbf{E}[\beta_0]$

$$P(\beta_0 \geq c) \leq \exp\left[-\frac{c - \mathbf{E}[\beta_0]}{6} \log\left(1 + \frac{c - \mathbf{E}[\beta_0]}{3\lambda}\right)\right]$$

Dimension 2

- $\beta_0 \leq$ Nb of points in a MHC process

$$\mathbf{E}[\beta_0] \leq \lambda \frac{1 - e^{-\theta}}{\theta} = \tau$$

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Références I

V. de Silva and R. Ghrist.
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