Random Sampling Auctions

Abraham Flaxman

(joint work with Uri Feige,
Jason Hartline, Bobby Kleinberg)

Outline

* Theory of Auctions
  * Random Sampling Auction
  * Analysis of RSOP
  * Equal Revenue Distribution
  * Computer aided proof

Theory of Auctions

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* Goods

Theory of Auctions

* Goods
* Bidders

* Theory of Auctions

* Goods
* Bidders
* Private values

Theory of Auctions

* Goods
* Bidders
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* Lying bastards

Theory of Auctions

* Standard approach:

<u>Truthful Mechanism Design</u>

Theory of Auctions

* Standard approach:

Truthful Mechanism Design

Come up with a social choice function and a payment function for which each bidder is best off revealing true preferences.

Theory of Auctions (Truthful Mech.)

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* 2 bidders, 1 item, maximize social welfare.

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* Mandatory example:
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  * Give item to the higher bidder,
  * Charge lower price.

Theory of Digital Goods Auctions

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* n bidders, as many items as you want, maximize revenue.

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* Soln: Split bidders randomly, find price for one part, offer to other.

* Theory of Digital Goods Auctions

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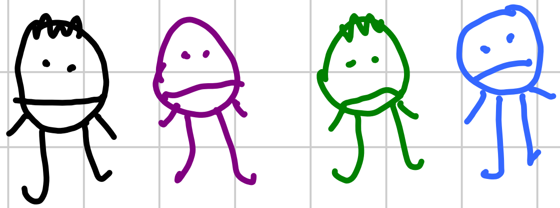

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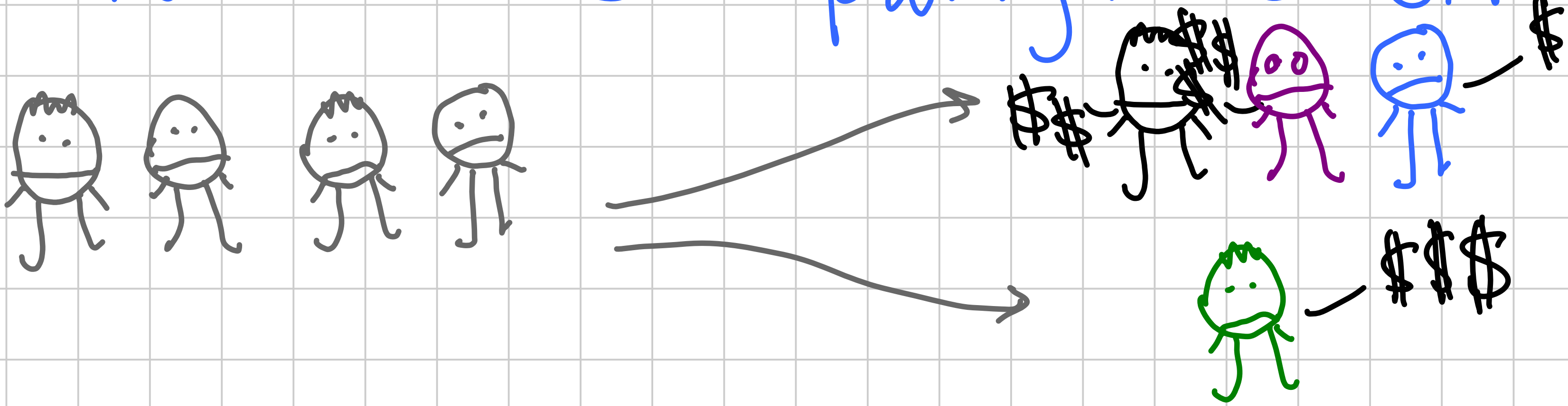

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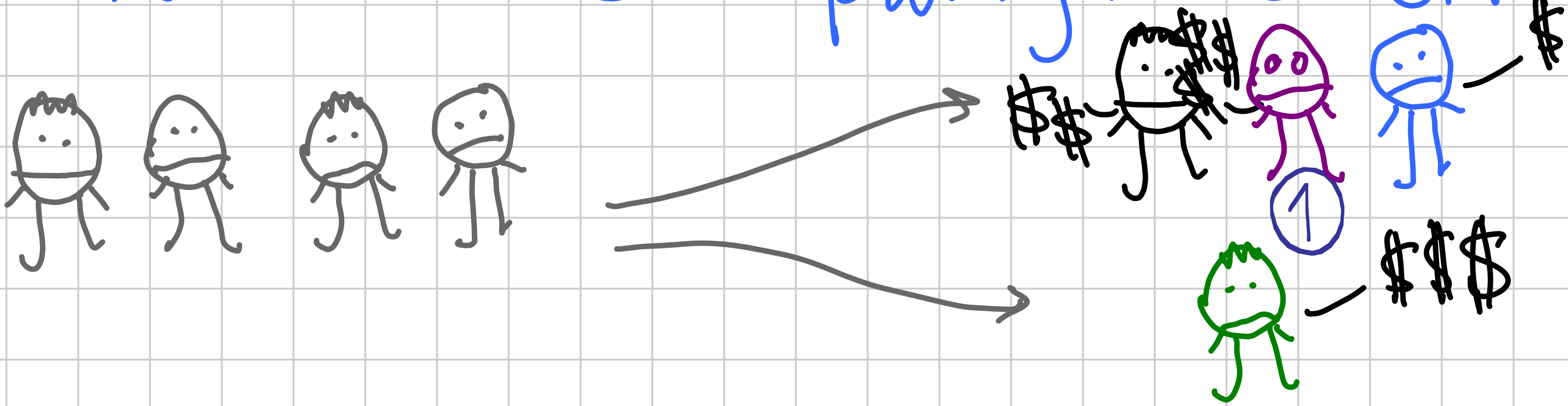

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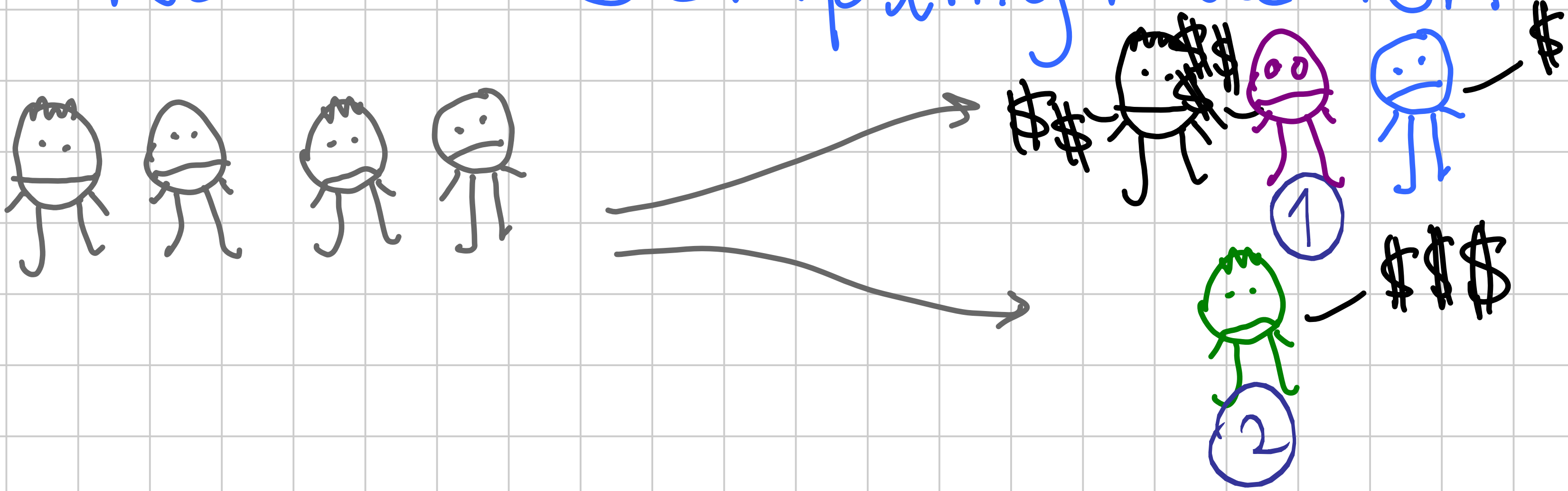

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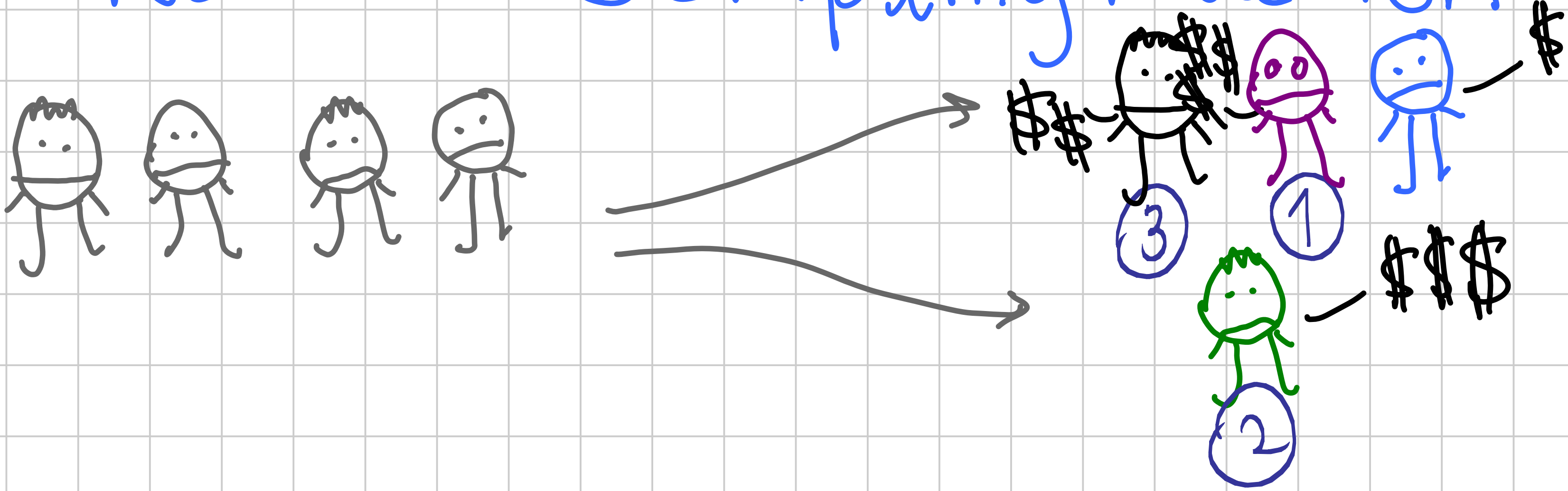

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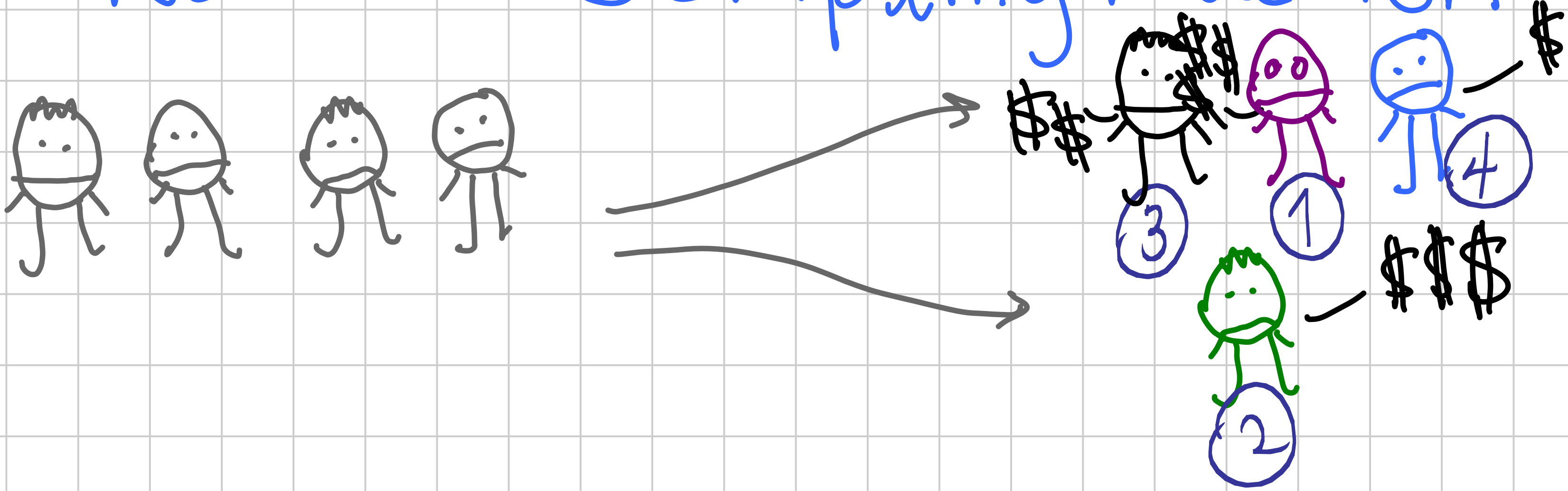

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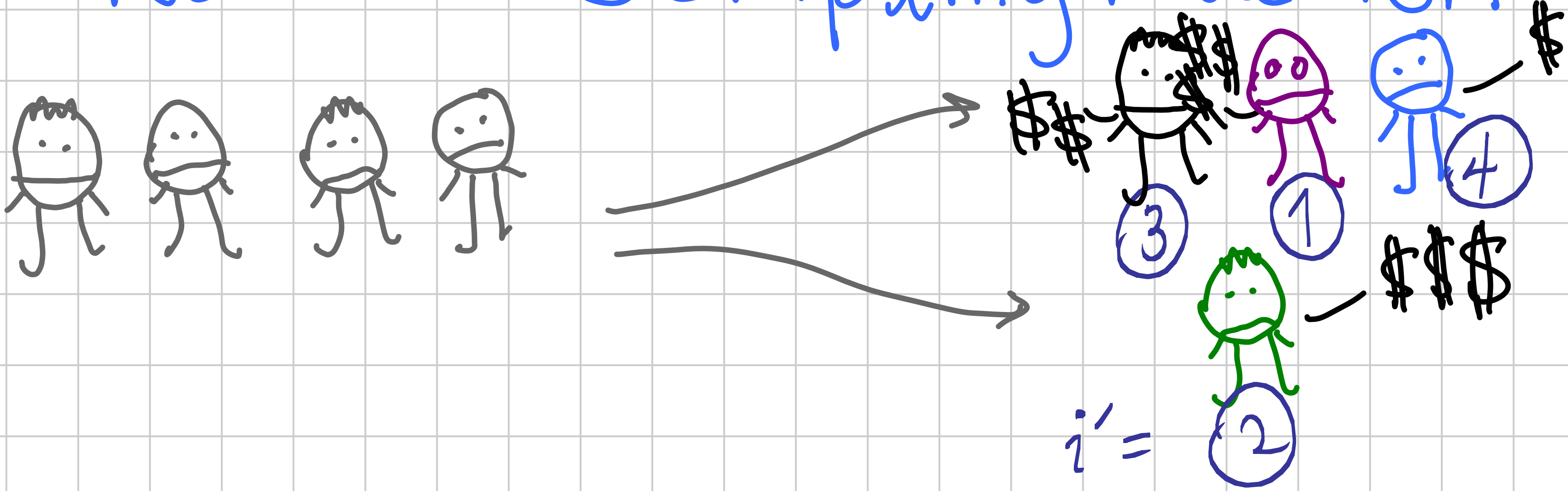

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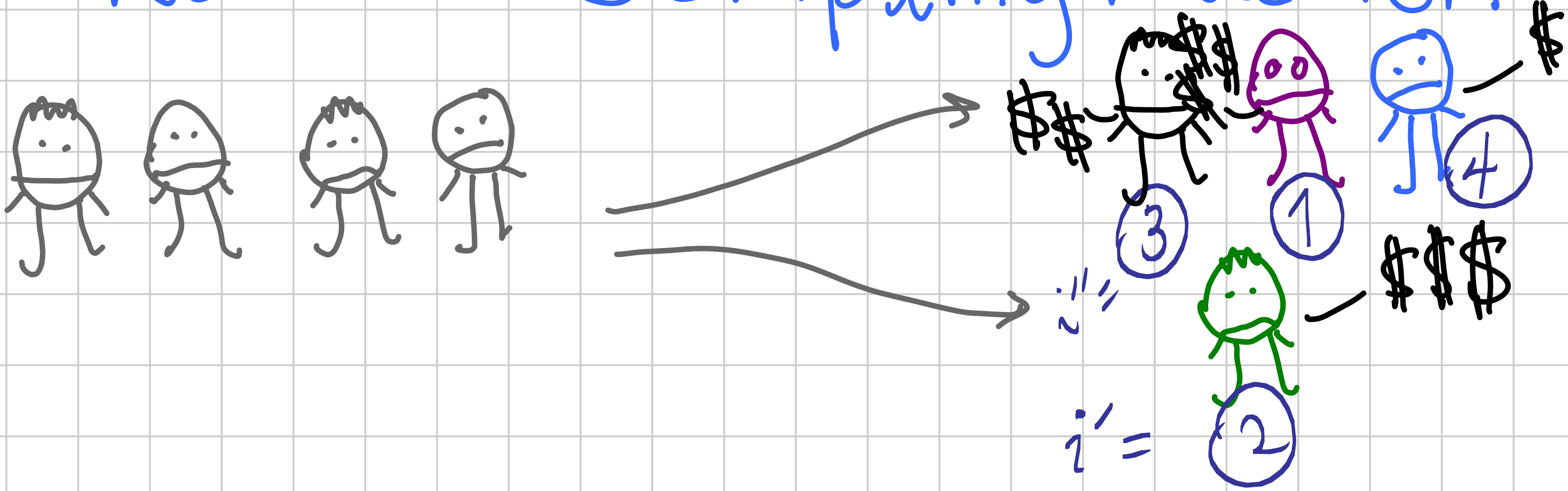

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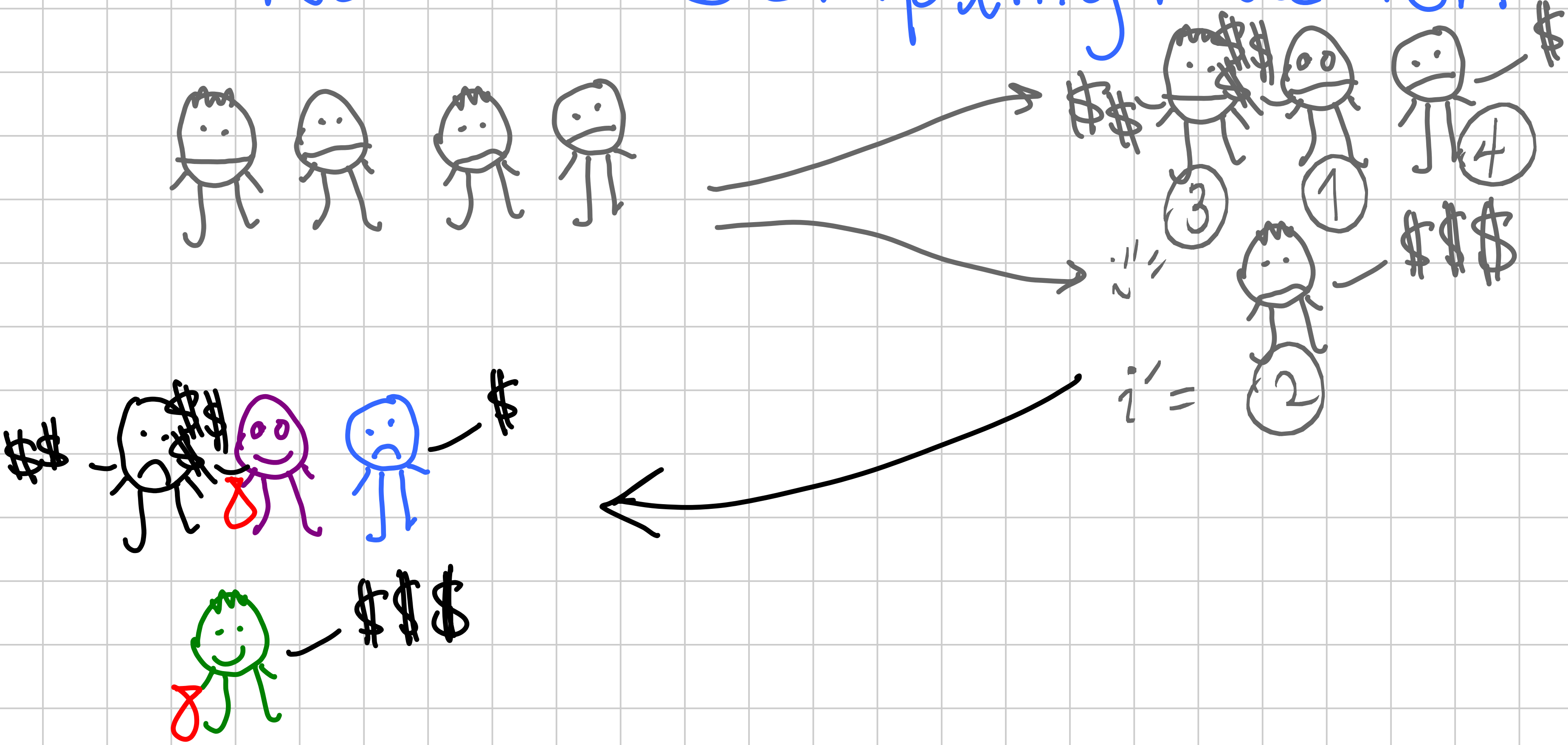

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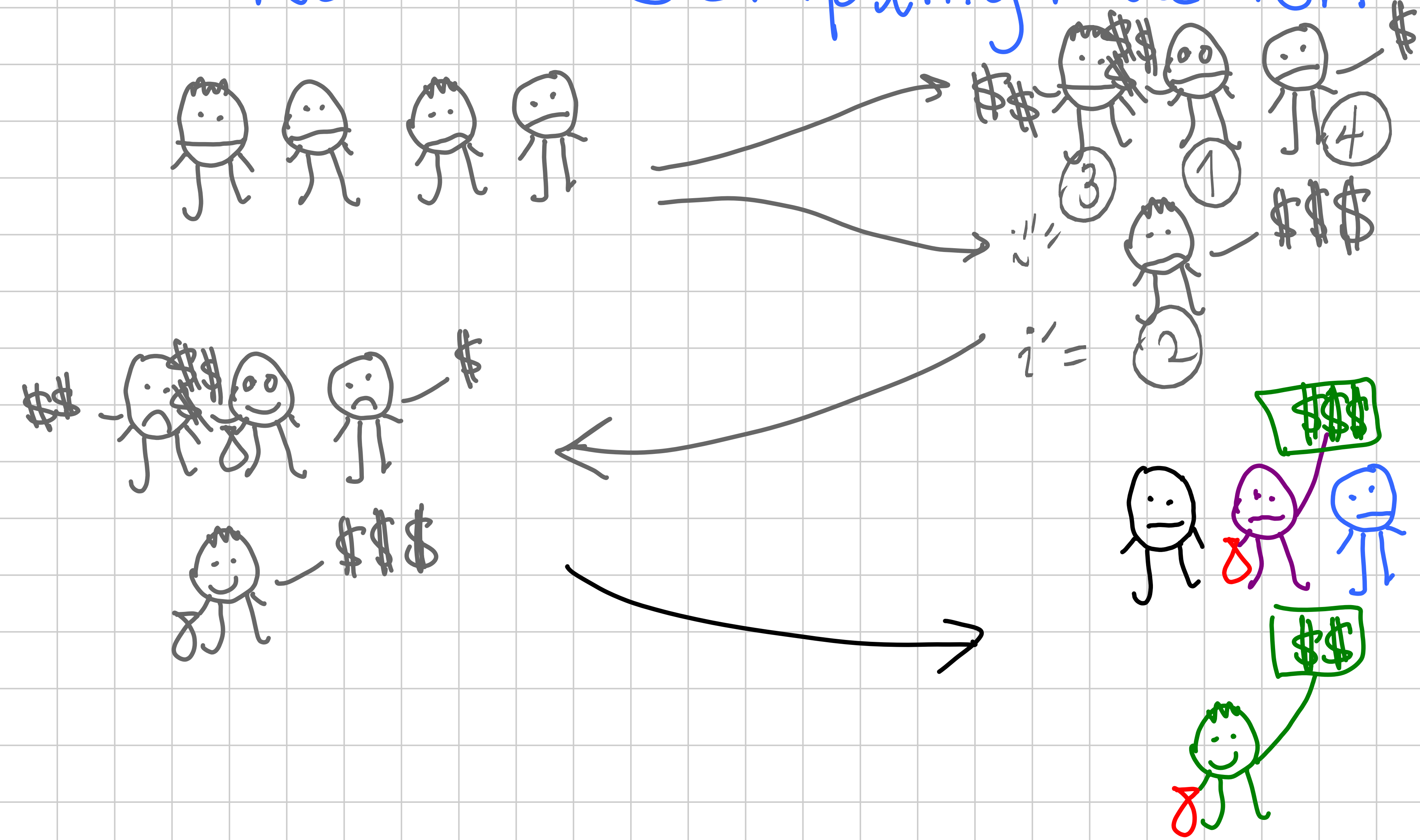

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  [Goldberg, Hartline, and Wright]

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Is it any good?

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(Compared to $f_2 = \max_{i \geq 2} i \cdot b_i$)

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[Goldberg, Hartline, and Wright]

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value of i-th highest bid

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[Goldberg, Hartline, and Wright]

Is it any good?

value of i-th highest bid

(Compared to $f_2 = \max_{i \geq 2} i \cdot b_i$)

* Previously shown to be very good in certain situations, within ×7600 for all bid vectors.

Theory of Digital Goods Auctions

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$$E[RS] \geq \frac{1}{15} f_2 .$$

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<u>Theorem:</u> For all $b_1 \geq b_2 \geq \dots \geq b_n$,

$$E[RS] \geq \frac{1}{15} f_2.$$

(The best value we can possibly have there is $\frac{1}{4}$.)

Theory of Digital Goods Auctions, Proof that $RS \geq \frac{1}{15} f_2$

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$$X_i = \begin{cases} 1 & \text{if partition splits bidder 1 from } i \\ 0 & \text{o.w.} \end{cases}$$

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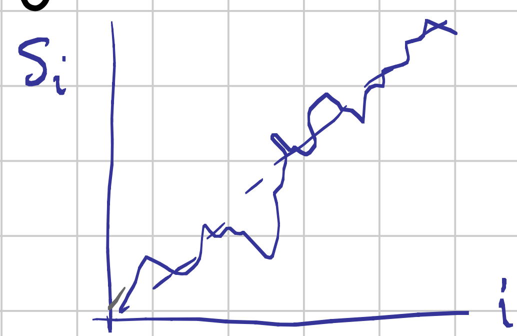

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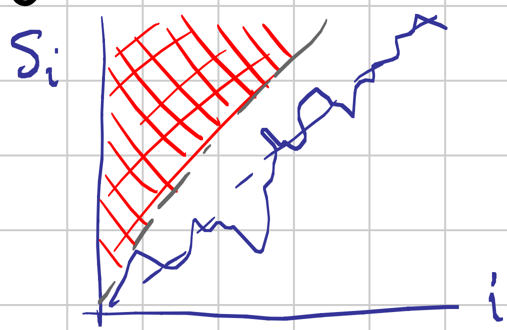

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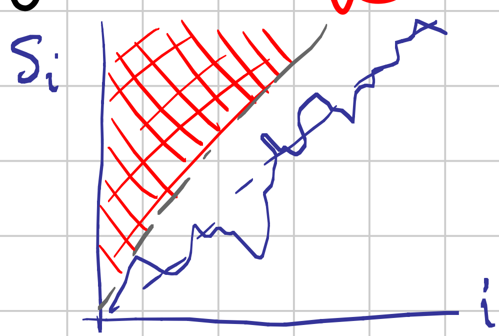

Digital Goods

✳ Theory of Digital Goods Auctions, Proof that $RS \geq \frac{1}{15} f_2$

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★ Theory of Digital Goods Auctions, Proof that $RS \geq \frac{1}{15} f_2$

So, $\mathbb{E}[RS] \geq \Pr[\mathcal{E} \wedge \mathcal{B}] \cdot \frac{1}{6} f_2$.

$\mathcal{E} = \{\forall i: S_i \leq \frac{3}{4} i\}$ $\mathcal{B} = \{S_{i'} \geq i'/2\}$

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Theory of Digital Goods Auctions, Proof that $RS \geq \frac{1}{15} f_2$

So, $\mathbb{E}[RS] \geq \mathbb{P}r[\mathcal{E} \wedge \mathcal{B}] \cdot \frac{1}{6} f_2$.

$$\mathbb{P}r[\mathcal{B} \wedge \mathcal{E}] \geq \mathbb{P}r[\mathcal{E}] - \mathbb{P}r[\overline{\mathcal{B}}]$$

$\mathcal{E} = \{\forall i : S_i \leq \frac{3}{4} i\}$ $\mathcal{B} = \{S_{i'} \geq i'/2\}$

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$$\text{So, } \mathbb{E}[RS] \geq \Pr[\mathcal{E} \wedge \mathcal{B}] \cdot \frac{1}{6} f_2 .$$

$$\Pr[\mathcal{B} \wedge \mathcal{E}] \geq \underbrace{\Pr[\mathcal{E}]}_{\text{Coming next!}} - \underset{\simeq .5}{\Pr[\overline{\mathcal{B}}]}$$

$$\mathcal{E} = \{\forall_i : S_i \leq \tfrac{3}{4} i\} \qquad \mathcal{B} = \{S_{i'} \geq i'/2\}$$

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$$\mathbb{P}r[\mathcal{B} \wedge \mathcal{E}] \geq \underbrace{\mathbb{P}r[\mathcal{E}]}_{\geq .9/2} - \underbrace{\mathbb{P}r[\overline{\mathcal{B}}]}_{\approx .5}$$

Coming next!

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Theory of Digital Goods Auctions, Calculating $\Pr[E]$

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$$\mathcal{E} = \left\{ \forall_i : S_i \leq \frac{3}{4} i \right\}$$

Theory of Digital Goods Auctions, Calculating $\Pr[\mathcal{E}]$

$$\mathcal{E} = \{\forall i : S_i \le \tfrac{3}{4} i\} \qquad \mathcal{E}_\alpha = \{\forall i : S_i \le \alpha \cdot i\}$$

Theory of Digital Goods Auctions, Calculating Pr[$\mathcal{E}$]

$$\mathcal{E} = \{\forall i : S_i \le \tfrac{3}{4} i\} \qquad \mathcal{E}_\alpha = \{\forall i : S_i \le \alpha \cdot i\}$$

For $\alpha = \frac{k}{k+1}$,

$$S_i \le \frac{k}{k+1} i \iff (k+1) S_i \le k \cdot i$$

$$\iff -S_i + k(i - S_i) \ge 0.$$

Theory of Digital Goods Auctions, Calculating $\Pr[\mathcal{E}]$

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Let $Z_i =$, asymmetric random walk

$$Z_i = \begin{cases} Z_{i-1} - 1, & \text{w. pr. } \frac{1}{2} \\ Z_{i-1} + k, & \text{w. pr. } \frac{1}{2} \end{cases}$$

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$$Pr[\mathcal{E}_{\frac{k}{k+1}}] = Pr[\forall i \;\; Z_i \geq 0 \mid Z_1 = k]$$

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$$=: 1 - p_k$$

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$$p_k = (p_0)^{k+1}$$

$$p_0 = \frac{1}{2} + \frac{1}{2} p_k$$

So p_0 is a root of $f(x) = 1 - 2x + x^{k+1}$

Theory of Digital Goods Auctions, Calculating Pr[E]

For $k=3$ (so $\alpha=\frac{k}{k+1}=\frac{3}{4}$), there is a closed form solution:

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Theory of Digital Goods Auctions, Calculating Pr[E]

For $k=3$ (so $\alpha = \frac{k}{k+1} = \frac{3}{4}$), there is a closed form solution:

$$p_0 = \frac{1}{3}\left[\left(17+3\sqrt{33}\right)^{1/3} - 1 - 2\left(17+3\sqrt{33}\right)^{-1/3}\right]$$

So p_0 is a root of $f(x) = 1 - 2x + x^{k+1}$

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and

$$\Pr[\mathcal{E}_{3/4}] = 1 - p_0^4$$

$$= 1 - \frac{1}{81}\left[\left(17+3\sqrt{33}\right)^{1/3} - 1 - 2\left(17+3\sqrt{33}\right)^{-1/3}\right]^4$$

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$$= 1 - \frac{1}{81}\left[\left(17+3\sqrt{33}\right)^{1/3} - 1 - 2\left(17+3\sqrt{33}\right)^{-1/3}\right]^4$$

$\simeq 0.912$ q.e.d.

★ Theory of Digital Goods Auctions, Equal revenue

Equal revenue input:

$$b_i = \frac{1}{i} \qquad i = 1, \dots, n$$

For $n = 2$, this has competitive ratio 4 (which we believe is max)

Theory of Digital Goods Auctions, Equal revenue

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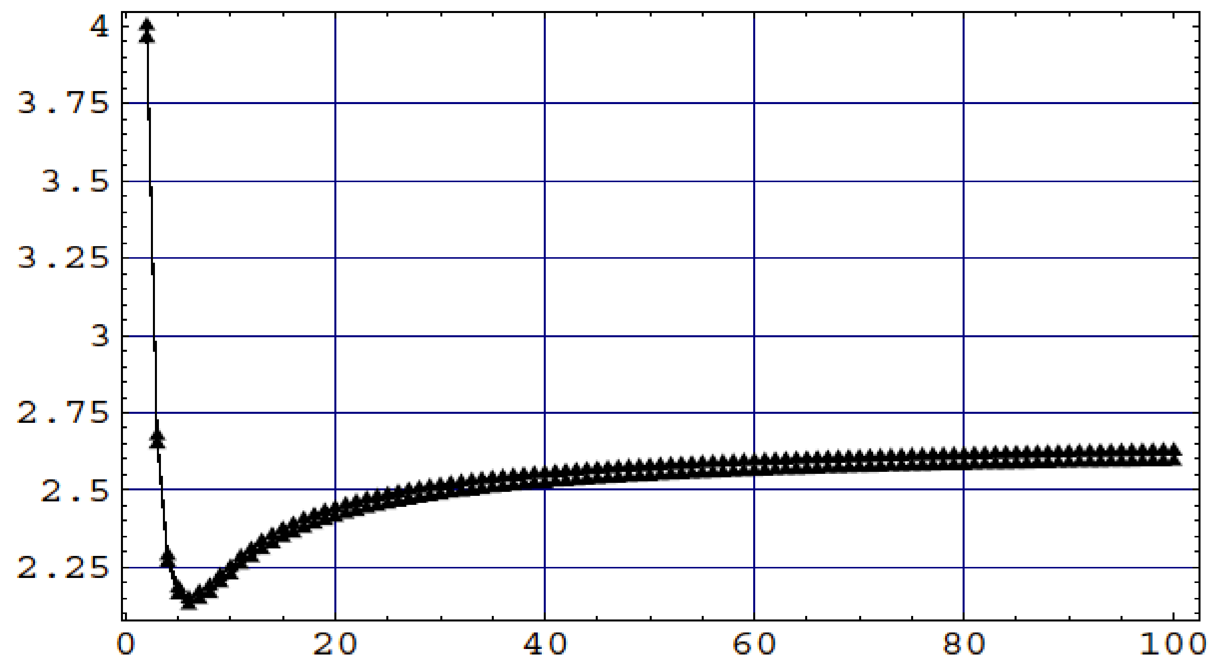

Fig. 1. Upper and lower bounds on $\mathcal{F}^{(2)} / \mathrm{E}[RS]$ when $N = 200$ for equal revenue input with $n = 2, \ldots, 100$.

Theory of Digital Goods Auctions, Equal revenue

For general n, define

$$\mathcal{E}_\alpha^n = \{\forall i \le n,\ S_i \le \alpha i\}.$$

Now, fix some integer N, and let

$$A_i^n = \mathcal{E}_{\frac{i}{N}}^n \cap \overline{\mathcal{E}_{\frac{i-1}{N}}^n}$$

Theory of Digital Goods Auctions, Equal revenue

Then

$$\mathbb{E}[RS] \geq \sum_{i=1}^{N-1} \mathbb{P}r[A_i^n]\left(1 - \frac{i}{N}\right)$$

which we can calculate exactly for specific values of n.

* Theory of Digital Goods Auctions, Computer Proof bounding $\mathcal{E}_\alpha$

$$p^\alpha(i,j) = \mathbb{P}\mathrm{r}\left[S_i = j \wedge \forall i' \leq i,\ S_{i'} \leq \alpha \cdot i'\right]$$

$$g^\alpha(i) = \sum_{j=0}^{i} p^\alpha(i,j)$$

* $$p^\alpha(i,j) = \begin{cases} \frac{1}{2}p^\alpha(i-1,j-1) + \frac{1}{2}p^\alpha(i-1,j), & \text{if } 0 \leq j \leq \alpha \cdot i; \\ 0, & \text{o.w.} \end{cases}$$

* $$\mathbb{P}\mathrm{r}\left[\overline{\mathcal{E}_\alpha}\right] \leq 1 - g^\alpha(i_0) + \sum_{i \geq i_0} \mathbb{P}\mathrm{r}\left[S_i \geq \alpha \cdot i\right]$$

$$\leq 1 - g^\alpha(i_0) + \sum_{i \geq i_0} e^{-(\alpha - \frac{1}{2})^2 i/3}$$

✱ Theory of Digital Goods Auctions, Computer Proof $\Pr[\mathcal{E}] \geq .912$

For reasonable values of n, it is possible to evaluate $\Pr[\mathcal{E}_\alpha^n]$. For example, for $\alpha = \frac{3}{4}$, $\Pr[\mathcal{E}_{\frac{3}{4}}^{200}]$ equals

$$\frac{22914483922452727752710576603653551719219315819721902777499}{25108406941546723055343157692830665664409421777856138051584}.$$

and so $\Pr[\mathcal{E}_{\frac{3}{4}}] \geq 0.912$

Conclusion

Theory of Auctions

- Random Sampling Auction
- Analysis of RSOP
- Computer aided proof
- Equal Revenue Distribution

* Open Questions

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* Is $E[RS] \geq \frac{1}{4} f_2$?

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* Can we say something more detailed in terms of bid vector?

Open Questions

* Is $E[RS] \geq \frac{1}{4} f_2$?

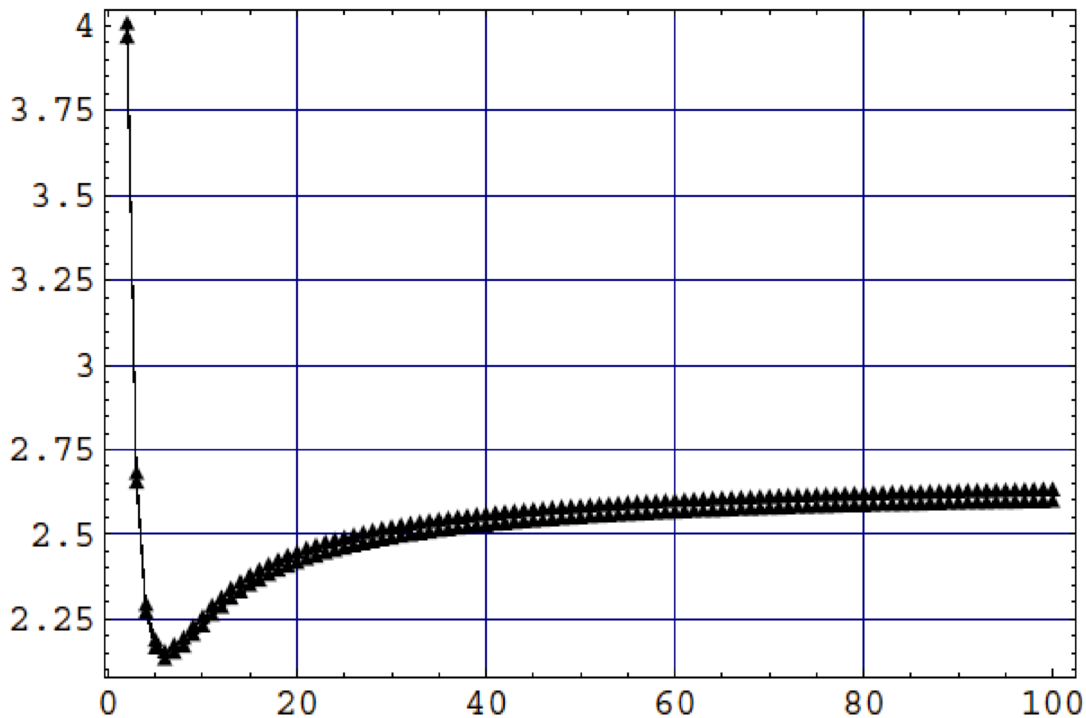

Fig. 1. Upper and lower bounds on $\mathcal{F}^{(2)} / \mathrm{E}[RS]$ when $N = 200$ for equal revenue input with $n = 2, \ldots, 100$.

Theory of Digital Goods Auctions, Equal revenue

$$\Pr[\mathcal{A}_i^n] \geq \Pr[\mathcal{E}_{\alpha_i}] - \Pr[\mathcal{E}_{\alpha_{i-1}}^{n_0}] \geq \Pr[\mathcal{E}_{\alpha_i}^{n_0}] - \frac{e^{-(\alpha_i - 1/2)^2 n_0/3}}{1 - e^{-(\alpha_i-1/2)^2/3}} - \Pr[\mathcal{E}_{a_{i-1}}^{n_0}].$$

So,

$$\mathrm{E}[RS] \geq (1 - \alpha_{i_0}) \Pr[\mathcal{E}_{\alpha_{i_0}}] + \sum_{i=i_0+1}^{N-1} \Pr[\mathcal{A}_i^{n_0}](1 - \alpha_i).$$

Taking $n_0 = 500$, $N = 100$, and $i_0 = 70$ (so $\alpha_{i_0} = 0.7$) and using the computer to prove bounds on the terms in this sum shows that for all $n \geq 500$, $\mathrm{E}[RS] \geq \mathcal{F}^{(2)}/3.6$. This, combined with the computer proof outlined previously for $n \leq n_0$, completes the proof showing that RSOP is 4-competitive on the equal revenue input.