First-Passage Percolation on a Width-2 Strip and the Path Cost in a VCG Auction

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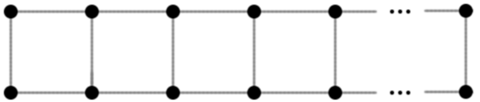
Outline

1. Introduction
   - What the title means
     - Width-2 strip
     - First-Passage Percolation
     - Path Cost in a VCG Auction
   - Fixed graphs with random edge weights
     - Minimum Spanning Tree
     - Minimum Perfect Matching
2. The width-2 strip
   - First-passage percolation
   - Path cost in a VCG auction

Width-2 Strip



- The *infinite width-2 strip*:
  - Vertex set is $\{0, 1\} \times \mathbb{Z}$
  - edges join vertices at ℓ_1 distance 1
- The *n-long strip* is the (finite) subgraph induced by $\{0, 1\} \times \{0, \ldots, n\}$.

Width-2 Strip

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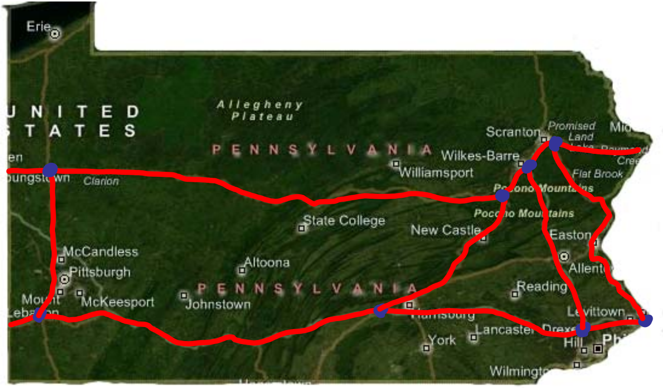

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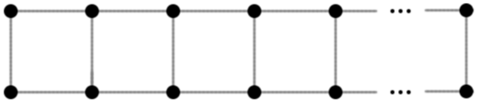

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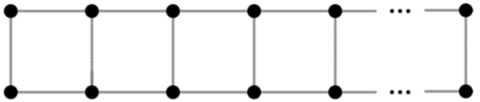

First-Passage Percolation:

- Models the time it takes a fluid to spread through a random medium.

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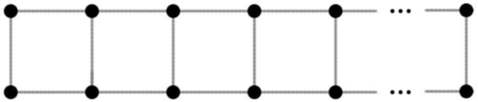

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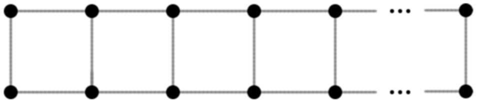

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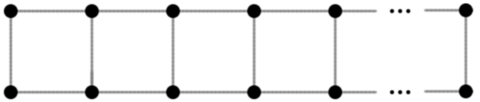

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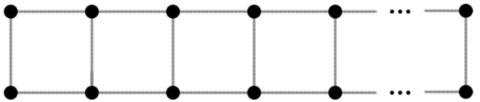
- The *time constant* is the limiting ratio of this length to the unweighted shortest path length n, as n tends to infinity.
- Introduced in Broadbent and Hammersley (1957) and Hammersley and Welsh (1965).

Path Cost in a VCG Auction



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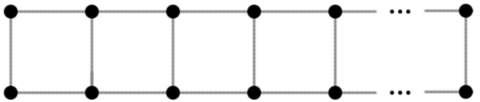

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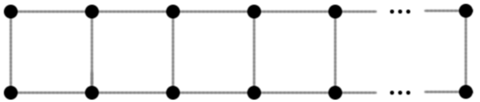

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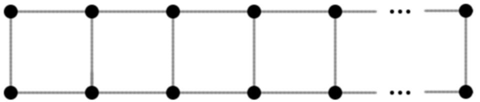

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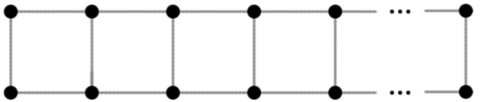

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- May require paying much more than the cost of the shortest path (more to say: Archer and Tardos (2002)).

Fixed graph with random edges weights

Today:
First passage percolation and path cost of VCG auction in the width-2 strip as specific examples of fixed graph with random edge weights.

Fixed graph with random edges weights, Ex 1: MST

Notable example of fixed graph with random edge weights:

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- Proof by studying a greedy algorithm for constructing MST [Frieze (1985)]

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- Rigorous proof of $\zeta(2)$ (*not* by analyzing known algorithm) [Aldous (2001)]

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Proof ingredients:

- An infinite object; fixed graph with random weights should converge to it; in this case, Poisson Infinite Weighted Tree (PWIT)
- A Recursive Distributional Equation (RDE) for a carefully chosen random variable of interest.
- A proof that the solution to the RDE on infinite object has something to do with the expectation for the finite object.

This present paper

Consider the present paper a simple example of that approach.

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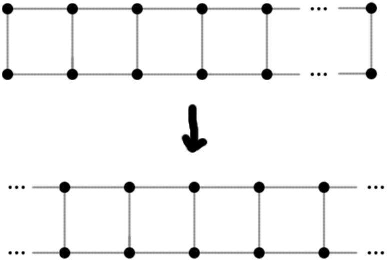
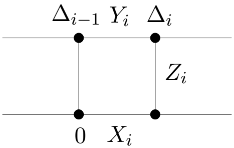
- Infinite analog of *n*-long width-2 strip is the infinite width-2 strip



First passage percolation in the width-2 strip

- Recursive distributional equations



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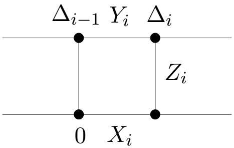

$$\ell(0,i) = \min\{\ell(0,i-1) + X_i, \quad \ell(1,i-1) + Y_i + Z_i\}$$
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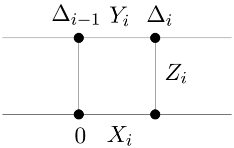

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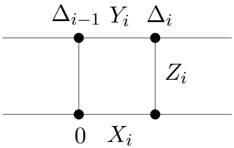

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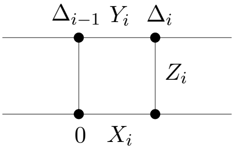
Better to consider $\Delta_i = \ell(1,i) - \ell(0,i)$.

First passage percolation in the width-2 strip

Recursive distributional equation for $\Delta_i = \ell(1, i) - \ell(0, i)$.



$$\Delta_i = \begin{cases} -Z_i, & \text{if } \Delta_{i-1} + X_i - Y_i < -Z_i; \\ \Delta_{i-1} + X_i - Y_i, & \text{if } \Delta_{i-1} + X_i - Y_i \in [-Z_i, Z_i]; \\ Z_i, & \text{if } \Delta_{i-1} + X_i - Y_i > Z_i. \end{cases}$$

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- $\lim_{n\to\infty} \frac{\mathbb{E}[\ell(0,n)]}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mathbb{E}[\gamma_i]}{n} = \lim_{n\to\infty} \mathbb{E}[\gamma_n]$.

What you get

If cost of edge $= \begin{cases} 0 & \text{w. pr. } p \\ 1 & \text{w. pr. } 1-p \end{cases}$ then shortest path from $(0,0)$ to $(n,0)$ tends to

$$\left(\frac{p^2(1+p)^2}{(3p^2+1)}\right) n.$$

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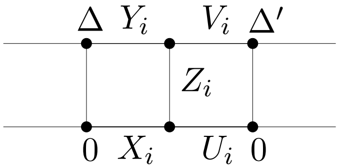
If cost of edge is uniform in $[0,1]$, then shortest path tends to $\approx (0.42\ldots)n$.

Path cost in a VCG auction

Same general approach can find the VCG cost of a path in the width-2 strip:



Results

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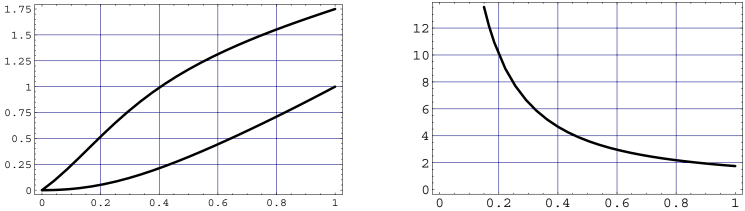

FIGURE 4. *Left:* VCG and usual shortest-path rates.
Right: Ratio of VCG cost to shortest-path cost.

Conclusion

Width-2 strip with random edge weights

- First-passage percolation
- VCG path auction

Extensions:

- Extend directly to Width-3 strip with no backtracking.
- Width-k strip?
- With backtracking?